SECTION 2.1
EXERCISE #37

2.1.37 Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be independent column vectors in $\mathbb{R}^n$ and let $C$ be an invertible $n \times n$ matrix. Prove that the vectors $C\vec{v}_1, C\vec{v}_2, \ldots, C\vec{v}_k$ are independent.

Proof

Suppose $r_1 C\vec{v}_1 + r_2 C\vec{v}_2 + \cdots + r_k C\vec{v}_k = \vec{0}$.

Then
$$C r_1 \vec{v}_1 + C r_2 \vec{v}_2 + \cdots + C r_k \vec{v}_k = \vec{0}$$
since scalars "Pull Through" (Theorem 2.3.A(7))

and
$$C(r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{v}_k) = \vec{0}$$
by the Distributive Law of Matrix Multiplication (Theorem 2.3.A(10)). Since $C$ is invertible, $C^{-1}$ exists and so
$$C^{-1} C(r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{v}_k) = C^{-1}\vec{0}$$
or
$$I(r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{v}_k) = \vec{0}$$
or
$$r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{v}_k = \vec{0}.$$
Since $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent then it must be that $r_1 = r_2 = \cdots = r_k = 0$ by Definition 2.1, "Linear Dependence and Independence." Therefore (again by Definition 2.1), $C\vec{v}_1, C\vec{v}_2, \ldots, C\vec{v}_k$ are linearly independent. ■