SECTION 2.2
EXERCISE #13

2.2.13 Let A be an $m \times n$ matrix and let $\vec{b}$ be an $n \times 1$ vector. Prove that the system of equations $A\vec{x} = \vec{b}$ has a solution for $\vec{x}$ if and only if $\text{rank}(A) = \text{rank}([A|\vec{b}])$.

Proof

Let the column vectors of A be $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$. Now $\text{rank}(A)$ is the dimension of the column space of A. By Theorem 2.1.A, some subset of $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ is a basis for the column space, say $\{\vec{a}_1', \vec{a}_2', \ldots, \vec{a}_{k'}\}$ where $k' = \text{rank}(A)$.

Suppose $A\vec{x} = \vec{b}$ has a solution $\vec{x}$. By Note 2.3.A, this implies that $\vec{b}$ is in the column space of A. So the column space is $sp(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n) = sp(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n, \vec{b})$.
So
$$\begin{aligned}\text{rank}(A) &= \dim(sp(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)) \\ &= \dim(sp(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n, \vec{b})) \\ &= \text{rank}([A|\vec{b}]).\end{aligned}$$

Suppose $\text{rank}(A) = \text{rank}([A|\vec{b}])$. Since
$$\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\} \subset \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n, \vec{b}\}$$
then $sp(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n) \subset sp(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n, \vec{b})$. The technique of finding a basis described in Theorem 2.2.A will produce the basis $\{\vec{a}_1', \vec{a}_2', \ldots, \vec{a}_{k'}\}$ for $sp(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)$. The same technique will produce a basis for $sp(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n, \vec{b})$ containing $\vec{a}_1', \vec{a}_2', \ldots, \vec{a}_{k'}$ and possibly vector $\vec{b}$. But since $\text{rank}(A) = \text{rank}([A|\vec{b}])$, then the second basis cannot also include $\vec{b}$ and so must be the same as the first basis. Since the bases are the same then the vector spaces are the same and $sp(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n) = sp(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n, \vec{b})$. ■