SECTION 2.2
EXERCISE #15

2.2.15 Exercise #14 has you prove that the column space of AC is contained in the column space of A. Is it true that the column space of AC is contained in the column space of C?

Solution

We take advantage of 0's and 1's. Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then the column space of C is

$$sp\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \mid y \in \mathbb{R} \right\}.$$ Next,

$$AC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$ and the column space of AC is

$$sp\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in \mathbb{R} \right\}.$$

So the column space of AC is NOT contained in the column space of C and the answer to the question is **NO.**

In fact, the only vector common to both column spaces is the zero vector $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Note The relationship of this exercise to Exercise #14 is resolved in Exercise #16 the answer to which is: "The row space of AC is contained in the row space of C." □