

SECTION 2.2
EXERCISE #17

2.2.17 Give an example of a 3×3 matrix A such that $\text{rank}(A) = 2$ and $\text{rank}(A^3) = 0$.

Solution

The only way we can have $\text{rank}(A^3) = 0$ is for A^3 to be a 3×3 zero matrix. This means that A is "nilpotent" (see Exercises 1.5.29, 1.5.30, and 1.5.31). We take

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{so that the column space of } A \text{ is } \text{sp}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \text{ and } \text{rank}(A) = 2.$$

Now

$$A^3 = A^2 A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so that $\text{rank}(A^3) = 0$, as desired. ■