SECTION 2.2
EXERCISE #5

2.2.5 Consider $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 2 \\ 0 & 2 & 1 & 1 \end{bmatrix}$.

(a) For rank(A), we reduce A:

$$A \xrightarrow[\text{WA}]{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 5/6 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \end{bmatrix} = H.$$

So H contains 3 pivots and hence 3 nonzero rows. So, by Note 2.2.A(1) a basis for the row space consists of the 3 (nonzero) rows of H. Hence the dimension of the row space is 3 and $\boxed{\text{rank}(A) = 3.}$

<u>NOTE</u> We can find the rank of matrix A by counting the pivots in (REF or RREF) H of A.

(b) A basis for the row space of A consists of the nonzero rows of H. So a basis is

$\boxed{\{[1, 0, 0, 5/6], [0, 1, 0, 1/3], [0, 0, 1, 1/3]\}.}$

Notice: Another basis (which doesn't use "fractions") is:

$\{[6, 0, 0, 5], [0, 3, 0, 1], [0, 0, 3, 1]\}$.

(c) A basis for the column space of A consists of the columns of A corresponding to the columns of H which contain pivots (see Note 2.2.A). So such a basis is

$$\boxed{\left\{ \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}.}$$

(d) Let's find a basis for the nullspace of A. Well, the nullspace of A is the set of all $\vec{x}$ such that $A\vec{x} = \vec{0}$. So consider the augmented matrix

$$[A|\vec{0}] = \left[\begin{array}{cccc|c} 0 & 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{array}\right] \underset{\text{WA}}{\xrightarrow{\text{RREF}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 5/6 & 0 \\ 0 & 1 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 1/3 & 0 \end{array}\right].$$

So, with $\vec{x} = [x_1, x_2, x_3, x_4]^T$ we need

$$\begin{aligned} x_1 \quad\quad + 5/6\, x_4 &= 0 \\ x_2 \quad + 2/3\, x_4 &= 0 \\ x_3 + 2/3\, x_4 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x_1 &= -5/6\, x_4 \\ x_2 &= -2/3\, x_4 \\ x_3 &= -2/3\, x_4 \\ x_4 &= \quad\; x_4 \end{aligned}.$$

Let $r = x_4/6$ (then $x_4 = 6r$) be a free variable. Then

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5r \\ -2r \\ -2r \\ 6r \end{bmatrix} = r \begin{bmatrix} -5 \\ -2 \\ -2 \\ 6 \end{bmatrix} \text{ where } r \in \mathbb{R}.$$

So the nullspace of A is $\text{sp}\left\{ \begin{bmatrix} -5 \\ -2 \\ -2 \\ 6 \end{bmatrix} \right\}$

and a basis for the nullspace is

$$\boxed{\left\{ \begin{bmatrix} -5 \\ -2 \\ -2 \\ 6 \end{bmatrix} \right\}}.$$