SECTION 2.3
NUMBERS 17 & 25

2.3.17 Consider the linear transformation
$T([x_1, x_2, x_3]) = [x_1 - x_2 + 3x_3, x_1 + x_2 + x_3, x_1]$.
Find the standard matrix representation.

<u>Solution</u>

To find the standard matrix representation, A_T, of T we apply T to the standard basis vectors for $\mathbb{R}^3$:

$T(\hat{i}) = T([1,0,0]) = [1-0+3(0), 1+0+0, 1]$
$= [1, 1, 1]$,
$T(\hat{j}) = T([0,1,0]) = [-1, 1, 0]$, and
$T(\hat{k}) = T([0,0,1]) = [3, 1, 0]$.

So, A_T has as its columns $T(\hat{i})^T$, $T(\hat{j})^T$ and $T(\hat{k})^T$, so

$$\boxed{A_T = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}}.$$

2.3.25 Is the above transformation invertible?

<u>Solution</u>

Well, T is an invertible linear transformation if and only if A_T is an invertible matrix (see Theorem 2.3.C). So consider

$$[A_T \mid I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{RREF}]{\text{We}}$$

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$$\underset{\text{RREF}}{\overset{\text{Wd}}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/4 & 3/4 & -1/2 \\ 1 & 0 & 1 & 1/4 & 1/4 & -1/2 \end{array}\right]$$

and so $A_T^{-1} = \frac{1}{4}\begin{bmatrix} 0 & 0 & 4 \\ -1 & 3 & -2 \\ 1 & 1 & -2 \end{bmatrix}$.

Therefore, $\boxed{\text{YES } T \text{ is invertible.}}$

A formula for T^{-1} is given by

$$A_T^{-1}\vec{x} = \frac{1}{4}\begin{bmatrix} 0 & 0 & 4 \\ -1 & 3 & -2 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4x_3 \\ -x_1 + 3x_2 - 2x_3 \\ x_1 + x_2 - 2x_3 \end{bmatrix},$$

and so

$$\boxed{T^{-1}([x_1, x_2, x_3]) = \frac{1}{4}\left[4x_3, -x_1 + 3x_2 - 2x_3, x_1 + x_2 - 2x_3\right]}$$

□