2.3.19 If $T : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $T([x_1, x_2]) = [2x_1 + x_2, x_1, x_1 - x_2]$ and $T' : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T'([x_1, x_2, x_3]) = [x_1 - x_2 + x_3, x_1 + x_2]$, then find the standard matrix representation for the linear transformation $T' \circ T$ that carries $\mathbb{R}^2$ into $\mathbb{R}^2$. Find a formula for $(T' \circ T)([x_1, x_2])$.

Solution

First, we find standard matrix representations of T and T' using Corollary 2.3.A. So we apply the transformations to the standard basis vectors and the output determines the columns of the matrices. So $T([1,0]) = [2,1,1]$ and $T([0,1]) = [1,0,-1]$, and $T([1,0,0]) = [1,1]$, $T([0,1,0]) = [-1,1]$, $T([0,0,1]) = [1,0]$; hence

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \text{ and } A' = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

By Theorem 2.3.B, the standard matrix representation of $T' \circ T$ is

$$A'A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}}.$$

A formula for $T' \circ T$ is given by

$$(T' \circ T)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 3x_1 + x_2 \end{bmatrix},$$

or $\boxed{(T' \circ T)[x_1, x_2] = [2x_1, 3x_1 + x_2]}$. □