SECTION 2.3
EXERCISE #33

2.3.33 The purpose of this exercise is to show that the reduced row-echelon form of a matrix is unique. Let A be an $m \times n$ matrix with row-echelon form H, and let V be the row space of A (and thus of H). Let $W_k = sp(\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_k)$ be the subspace of $\mathbb{R}^n$ generated by the first k rows of the $n \times n$ identity matrix. Consider $T_k : V \to W_k$ defined by

$$T_k([x_1, x_2, \ldots, x_n]) = [x_1, x_2, \ldots, x_k, 0, \ldots, 0].$$

(a) Prove that T_k is a linear transformation of V into W_k and that $T_k[V] = \{T_k(\vec{v}) \mid \vec{v} \in V\}$ is a subspace of W_k.

Proof

Let $\vec{u} = [u_1, u_2, \ldots, u_n]$, $\vec{v} = [v_1, v_2, \ldots, v_n] \in V$ and $r \in \mathbb{R}$. Then

$$\begin{aligned}
T_k(\vec{u} + \vec{v}) &= T_k([u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]) \\
&= T_k([u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n]) \\
&= [u_1 + v_1, u_2 + v_2, \ldots, u_k + v_k, 0, 0, \ldots, 0] \\
&= [u_1, u_2, \ldots, u_k, 0, \ldots, 0] + [v_1, v_2, \ldots, v_k, 0, \ldots, 0] \\
&= T_k([u_1, u_2, \ldots, u_n]) + T_k([v_1, v_2, \ldots, v_n]) \\
&= T_k(\vec{u}) + T_k(\vec{v})
\end{aligned}$$

and so T_k preserves addition. Next,

$$\begin{aligned}
T_k(r\vec{u}) &= T_k(r[u_1, u_2, \ldots, u_n]) = T_k([ru_1, ru_2, \ldots, ru_n]) \\
&= [ru_1, ru_2, \ldots, ru_k, 0, \ldots, 0] = r[u_1, u_2, \ldots, u_k, 0, \ldots, 0] \\
&= r\, T_k([u_1, u_2, \ldots, u_n]) = r\, T_k(\vec{u})
\end{aligned}$$

and so T_k preserves scalar multiplication. Hence, by Definition 2.3, "Linear Transformation," T_k is linear.

By Theorem 2.3.A(2), since V is a subspace of $\mathbb{R}^n$ (V is the row space of $m \times n$ matrix A), then $T_k[V]$ is a subspace of W_k.

(b) Show that if $T_k[V]$ has dimension d_k then for each $j < n$ we have either $d_{j+1} = d_j$ or $d_{j+1} = d_j + 1$.

Solution

Notice that

$$\begin{aligned} T_j[V] &= \{[v_1, v_2, \ldots, v_j, 0, \ldots, 0] \mid \vec{v} = [v_1, v_2, \ldots, v_n] \in V\} \\ &\subset \{[v_1, v_2, \ldots, v_j, v_{j+1}, 0, \ldots, 0] \mid \vec{v} = [v_1, v_2, \ldots, v_n] \in V\} \\ &= T_{j+1}[V] \text{ for } j = 1, 2, \ldots, n-1, \end{aligned}$$

so $\dim(T_j[V]) \le \dim(T_{j+1}[V])$ or $d_j \le d_{j+1}$. Since $T_j[V]$ is a subspace of W_j then

$d_j = \dim(T_j[V]) \le \dim(W_j) = j$.

Now $d_1 = 0$ if and only the first component of every row vector of A (and so every vector in the row space V of matrix A) is 0. In this case the first column of row-echelon form H must consist only of 0's and so column 1 of H has no pivot. Next, $d_1 = 1$ if and only if the first component of some row vector of A is nonzero. In this case, column 1 of H contains a pivot (and since H is in row-echelon form, the pivot, if it exists, is in the (1, 1) position in H). If $d_1 = 1$ then denote the first row of H as $\vec{h}_1$ so that $\{T_1(\vec{h}_1)\}$ is a basis for $T_1[V]$. If $d_1 = 0$ then let $\vec{h}_1$ denote the zero vector, $\vec{h}_1 = \vec{0}$.

We now consider d_2. If $d_1 = 0$ then we repeat the above argument to conclude that $d_2 = 0$ if and only if column 2 of H contains a pivot (and since H is in row-echelon form, the pivot, if it exists, is in the $(1,2)$ position of H). If column 2 of H does not contain a pivot then $d_1 = d_2 = 0$. If column 2 of H does contain a pivot then $d_2 = 1$ and so $d_2 = d_1 + 1$. If $d_2 = 1$ then denote the first row of H as $\vec{h}_2$ so that $\{T_2(\vec{h}_2)\}$ is a basis for $T_2[V]$. If $d_2 = 0$ then let $\vec{h}_2$ denote the zero vector, $\vec{h}_2 = \vec{0}$.

If $d_1 = 1$ then again consider the second column of H. If the second column of H contains a pivot (which, in this case, is in the $(2,2)$ position of H) then $d_2 = 2$ (so that $d_2 = d_1 + 1$) and a basis for $T_2[V]$ is $\{T_2(\vec{h}_1), T_2(\vec{h}_2)\}$ where $\vec{h}_2$ denotes the second row of H. If the second column of H does not contain a pivot then (in this case) $d_2 = \dim(T_2[V]) = \dim(T_1[V]) = d_1$ and we set $\vec{h}_2 = \vec{0}$.

Repeating this process, a basis for $T_j[V]$ is given by $\{T_j(\vec{h}_1), T_j(\vec{h}_2), \ldots, T_j(\vec{h}_j)\} \setminus \{\vec{0}\}$ and this basis consists of d_j vectors. We consider column $j+1$ of H. If this column contains a pivot then we let $\vec{h}_{j+1}$ denote the <u>row</u> of H containing the pivot. If column $j+1$ of H does not contain a pivot then let $\vec{h}_{j+1}$ denote the zero vector $\vec{h}_{j+1} = \vec{0}$. Then a basis for $T_{j+1}[V]$ is $\{T_{j+1}(\vec{h}_1), T_{j+1}(\vec{h}_2), \ldots, T_{j+1}(\vec{h}_{j+1})\}$ and this basis consists of d_{j+1} vectors. →

If $\vec{h}_{j+1} = \vec{0}$ then $d_{j+1} = d_j$ and if $\vec{h}_{j+1} \neq \vec{0}$ then $d_{j+1} = d_j + 1$, as claimed. □

(c) Assume that A has four columns. Referring to part (b), suppose $d_1 = d_2 = 1$ and $d_3 = d_4 = 2$. Find the number of pivots in H and give the location of each.

Solution

Let $H = [h_{ij}]$. Since $d_1 = 1$ then h_{11} must be a pivot in H and $h_{11} \neq 0$. In this solution we set each pivot equal to 1. As shown in the solution to (b), $d_{j+1} = d_j$ if and only if column $j+1$ does not contain a pivot.
So we have

$$H = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(assuming that H is square).

Our entries of "2" can be replaced with any other number (and the pivots of "2" can be replaced with any other nonzero number). Notice that, for this example, a basis for $T_1[V]$ is $\{T_1([1,2,2,2])\} = \{[1,0,0,0]\}$, a basis for $T_2[V]$ is $\{T_2([1,2,2,2])\} = \{[1,2,0,0]\}$, a basis for $T_3[V]$ is $\{T_3([1,2,2,2]), T_3([0,0,1,2])\}$ $= \{[1,2,2,0], [0,0,1,0]\}$, and a basis for $T_4[V]$ is $\{T_4([1,2,2,2]), T_4([0,0,1,2])\} = \{[1,2,2,2], [0,0,1,2]\}$.

□

(d) Repeat part (c) for the case where A has six columns and $d_1 = 1$, $d_2 = d_3 = d_4 = 2$, and $d_5 = d_6 = 3$.

Solution

Let $H = [h_{ij}]$. Since $d_1 = 1$ then h_{11} must be a pivot in H and $h_{11} \neq 0$. As in the solution to part (c), we set all pivots in H to 1 and make all nonzero nonpivots 2. Since the maximum d_j value is 3 then we only need 3 rows in H. As shown in the solution to part (b), $d_{j+1} = d_j$ if and only if column $j+1$ does not contain a pivot. So we have:

$$H = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

Our entries of "2" can be replaced with any other number (and the pivots of "1" can be replaced with any nonzero number). □

(e) Argue that, for any matrix A, the number of pivots and the location of each pivot in any row-echelon form of A is always the same.

Solution

As seen in the solution to part (b), $d_{j+1} = d_j + 1$ if and only if column $j+1$ of row-echelon form H (where $H \sim A$) contains a pivot; this holds for $j = 0, 1, \ldots, n-1$ provided we take $d_0 = 0$. So the number of pivots in H is the number of distinct d_j, $\operatorname{rank}(H) = |\{d_1, d_2, \ldots, d_n\}|$. The pivot in the kth row of H occurs in the jth column of H if and only if d_j is the kth distinct positive integer in the list $d_1, d_2, \ldots, d_n$. [To better understand this

notice that in part (d) where $d_1 = 1$, $d_2 = d_3 = d_4 = 2$, and $d_5 = d_6 = 3$, the distinct d_i appear in order as d_1, d_2, d_5 (these correspond to $k = 1, 2, 3$ and $j = 1, 2, 5$, respectively) and the pivots of H occur in positions $(k, j) \in \{(1,1), (2,2), (3,5)\}$.]
The numbers d_i depend only on matrix A (this is the reason for introducing the row space V of matrix A and the mappings $T_k : V \to W_k$ in part (a); to produce the d_i from matrix A). So the number of and location of the pivots in H, where H is <u>any</u> row-echelon equivalent of matrix A, depend only on matrix A. That is, the number of pivots and location of each pivot in any row-echelon form of A is always the same, as claimed. □

(f) Show that the reduced row-echelon form of a matrix A is unique.

<u>Solution</u>

As seen in part (e), if H is a row-echelon form of A then the number and position of the pivots of H is uniquely determined. We can produce reduced row-echelon K where $K \sim H \sim A$ by dividing each pivot-containing row of H by the pivot value in H (thus making all pivots 1 in K) and by then eliminating nonzero entries above pivots with row addition. We need to show that K is uniquely determined by A. Consider row k of matrix K where $1 \le k \le \text{rank}(K) = \text{rank}(A)$ (by Note 2.2.A; notice that the kth row of K is necessarily all 0's when $k > \text{rank}(K) = \text{rank}(A)$). Then row k contains

a pivot of K, say in column j of matrix K. Now suppose $\vec{r} \in \mathbb{R}^n$ is a row vector with entries of 0 in all components corresponding to columns of H containing pivots except for the jth component where the entry is 1 (the entries in components corresponding to non-pivot-containing columns of H can be anything; so there are potentially an infinite number of choices for $\vec{r}$).
[For example, with $k=2$ in part (d), we would have $\vec{r} = [0, 1, X, X, 0, X]$ based on matrix H of part (d) since row 2 of H has a pivot in column 2 of row 2 and H also has pivots in columns 1 and 5.]
So row k of matrix K is of the form of vector $\vec{r}$. But row k of matrix K is in the row space V of matrix A by Note 2.2.A(1). So we now show that there is a unique element of V that is of the form $\vec{r}$.
If $\vec{r} \in V$ then $\vec{r}$ must be a linear combination of the nonzero rows of K since these rows form a basis for V (again, by Note 2.2.A(2)).
If column l of K contains a pivot where $l \neq j$ then the lth entry of $\vec{r}$ is 0. But since K is in reduced row-echelon form then the lth entry of rows of K is 0 except the row containing the pivot. So when writing $\vec{r}$ as a linear combination of the nonzero rows of K then all coefficients must be 0 except for the coefficient of row k. Since row k has a pivot of 1 in the jth component and so does vector $\vec{r}$ then the coefficient of

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row k must be 1. That is, if $\vec{r}$ is as described and $\vec{r} \in V$ (the column space of A) then $\vec{r} = 0\vec{r}_1 + 0\vec{r}_2 + \cdots + 0\vec{r}_{k-1} + 1\vec{r}_k + 0\vec{r}_{k+1} + \cdots + 0\vec{r}_n$ (where $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$ are the nonzero rows of K). So there is a unique choice for the kth row of matrix K. Since this holds for all $1 \leq k \leq \text{rank}(K)$ then the reduced row-echelon form K of matrix A is unique, as claimed. $\square$