SECTION 2.4
EXERCISE #17

2.4.17 Use the familiar equation that describes the dot product $\vec{u}\cdot\vec{v}$ geometrically to prove that if a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ preserves both length and angle, then it also preserves dot products.

Proof

The "familiar equation" is that $\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$ where θ is the angle between vectors $\vec{u}$ and $\vec{v}$ (see Section 2.2, "The Norm and Dot Product"). If linear transformation T preserves length then $\|T(\vec{u})\| = \|\vec{u}\|$ and $\|T(\vec{v})\| = \|\vec{v}\|$, and if T preserves angles then the angle between $T(\vec{u})$ and $T(\vec{v})$ is also θ. So

$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$ by the "familiar equation"

$= \|T(\vec{u})\|\,\|T(\vec{v})\|\cos\theta$ since T preserves length

$= \|T(\vec{u})\|\,\|T(\vec{v})\|\cos\theta$ since T preserves angles (so we "replace" θ with θ)

$= T(\vec{u})\cdot T(\vec{v})$ by the "familiar equation".

So T also preserves dot products. ∎