SECTION 2.4
EXERCISE #19

2.4.19 Express both the length of a vector $\vec{v} \in \mathbb{R}^2$ and the angle between two nonzero vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$ in terms of the dot product only. (From this we may conclude that if a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ preserves the dot product, then it preserves length and angle. This is the converse of Exercise #17.)

Solution

The length of $\vec{v}$ is $\boxed{\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}.}$

The angle θ between $\vec{u}$ and $\vec{v}$ satisfies $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$ by the "familiar equation from Section 2.2, "The Norm and Dot Product" and from Exercise # 17. Solving for θ we have
$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

or
$$\boxed{\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}\right) = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\sqrt{(\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v})}}\right).} \quad \square$$