SECTION 2.4

EXERCISE #21

2.4.21 Prove that the two column vectors of a 2×2 matrix A are orthogonal unit vectors if and only if $A^T A = \mathcal{I}$. Demonstrate that the matrix representations for the rigid motions in Examples 1 and 2 satisfy this condition.

<u>Proof</u>

Let the column vectors of A be $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Then $A = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$ and $A^T = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$.

So

$$A^T A = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} = \begin{bmatrix} u_1 u_1 + u_2 u_2 & u_1 v_1 + u_2 v_2 \\ u_1 v_1 + u_2 v_2 & v_1 v_1 + v_2 v_2 \end{bmatrix}$$

$$= \begin{bmatrix} \vec{u} \cdot \vec{u} & \vec{u} \cdot \vec{v} \\ \vec{u} \cdot \vec{v} & \vec{v} \cdot \vec{v} \end{bmatrix}.$$

Hence $A^T A = \mathcal{I}$ if and only if $\vec{u} \cdot \vec{u} = 1$ and $\vec{v} \cdot \vec{v} = 1$ (i.e., $\|\vec{u}\| = 1$ and $\|\vec{v}\| = 1$), and $\vec{u} \cdot \vec{v} = 0$ (i.e., $\vec{u}$ and $\vec{v}$ are orthogonal), as claimed. ∎

<u>Note</u> This result holds for $n \times n$ matrices as well and motivates the definition of <u>orthogonal matrix</u> in Section 6.3.

<u>Solution</u>

The rigid motion of Example 1 (see pages 156-57) is $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. With the columns as $\vec{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ we have

$\vec{u} \cdot \vec{u} = \cos^2\theta + \sin^2\theta = 1$, $\vec{v} \cdot \vec{v} = (-\sin\theta)^2 + \cos^2\theta = 1$,

and $\vec{u} \cdot \vec{v} = \cos\theta(-\sin\theta) + \sin\theta\cos\theta = 0$ and so A satisfies "this condition" (so $A^T A = \mathcal{I}$).

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SECTION 2.4

EXERCISE #21 (continued)

The rigid motion of Example 2 is described by $A = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$. With the columns as

$\vec{u} = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$ we have

$$\vec{u} \cdot \vec{u} = \left(\frac{-3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{9+16}{25} = \frac{25}{25} = 1,$$

$$\vec{v} \cdot \vec{v} = \left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = \frac{16+9}{25} = \frac{25}{25} = 1, \text{ and}$$

$$\vec{u} \cdot \vec{v} = \left(\frac{-3}{5}\right)\left(\frac{4}{5}\right) + \left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{-12+12}{25} = \frac{0}{25} = 0.$$

So $\|\vec{u}\| = \|\vec{v}\| = 1$, $\vec{u}$ is perpendicular to $\vec{v}$ and so A satisfies "this condition" (so $A^T A = I$). $\square$