SECTION 2.5
EXERCISE #38

2.5.38 Solve and express the solution as a k-flat:

$$x_1 - 3x_2 + 2x_3 - x_4 = 8$$
$$3x_1 - 7x_2 + x_4 = 0.$$

<u>Solution</u>

Well, consider the augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & -3 & 2 & -1 & 8 \\ 3 & -7 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{cccc|c} 1 & -3 & 2 & -1 & 8 \\ 0 & 2 & -6 & 4 & -24 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2/2} \left[\begin{array}{cccc|c} 1 & -3 & 2 & -1 & 8 \\ 0 & 1 & -3 & 2 & -12 \end{array}\right] \xrightarrow{R_1 \to R_1 + 3R_2} \left[\begin{array}{cccc|c} 1 & 0 & -7 & 5 & -28 \\ 0 & 1 & -3 & 2 & -12 \end{array}\right].$$

So we consider the associated system of equations

$$x_1 \quad - 7x_3 + 5x_4 = -28 \quad \text{or} \quad x_1 = -28 + 7x_3 - 5x_4$$
$$x_2 - 3x_3 + 2x_4 = -12 \qquad x_2 = -12 + 3x_3 - 2x_4$$

or
$$\begin{aligned} x_1 &= -28 + 7x_3 - 5x_4 \\ x_2 &= -12 + 3x_3 - 2x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$
, letting $r = x_3$ and $s = x_4$ be free variables and then

$$\begin{aligned} x_1 &= -28 + 7r - 5s \\ x_2 &= -12 + 3r - 2s \\ x_3 &= r \\ x_4 &= s \end{aligned}$$

where $r, s \in \mathbb{R}$. In vector notation we have:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -28 \\ -12 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} 7 \\ 3 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -5 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$ where $r, s \in \mathbb{R}$.

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so $\vec{x}$ is in the solution set of the original system of equations if

$$\boxed{\vec{x} \in \vec{a} + W = \begin{bmatrix} -28 \\ -12 \\ 0 \\ 0 \end{bmatrix} + \mathrm{sp}\left(\begin{bmatrix} 7 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 0 \\ 2 \end{bmatrix} \right).}$$

Notice that this is a 2-flat in $\mathbb{R}^4$, so geometrically the solution set is a plane in $\mathbb{R}^4$! ☺