SECTION 2.5
NUMBER 9

2.5.9 Consider the line in $\mathbb{R}^2$ that is given by the equation $d_1 x_1 + d_2 x_2 = c$ for numbers $d_1, d_2, c \in \mathbb{R}$, where d_1 and d_2 are not both 0. Find the parametric equations of the line.

Solution

Notice that $d_1 x_1 + d_2 x_2 = c$ is equivalent to $x_2 = -(d_1/d_2)x_1 + c/d_2$ if $d_2 \neq 0$. This is a line of slope $m = -d_1/d_2$ (if $d_2 = 0$, we have a vertical line of the form $x_1 = c/d_1$). So if $d_2 \neq 0$, we can let $s = x_1$ be a parameter and then we have the parametric equations

$$x_1 = s$$
$$x_2 = -(d_1/d_2)s + c/d_2$$

Similarly, if $d_1 \neq 0$ then $x_1 = -(d_2/d_1)x_2 + c/d_1$ and we can let $s = x_2$ be a parameter to give the parametric equations

$$x_1 = -(d_2/d_1)s + c/d_1$$
$$x_2 = s$$

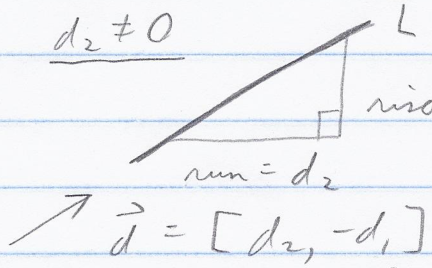
So the line is given parametrically as

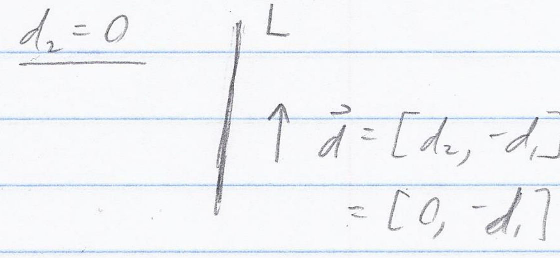
$$\boxed{\begin{aligned} x_1 &= \begin{cases} s & \text{if } d_2 \neq 0 \\ -(d_2/d_1)s + (c/d_1) & \text{if } d_2 = 0 \end{cases} \\ x_2 &= \begin{cases} -(d_1/d_2)s + (c/d_2) & \text{if } d_2 \neq 0 \\ s & \text{if } d_2 = 0 \end{cases} \end{aligned}}$$

This does not look like the answer in the back of the book. We can get that answer as follows.

SECTION 2.5
NUMBER 9 (continued)

We observed above that $d_1 x_1 + d_2 x_2 = c$ is a line of slope $-d_1/d_2$ if $d_2 \neq 0$ and is a vertical line if $d_2 = 0$. Notice that in either case, a direction vector of the line is $\vec{d} = [d_2, -d_1]$:





So we take $\vec{d} = [d_2, -d_1]$ for the direction vector and we need a point. To get the book's answer, we observe that (x_1, x_2) is a point on the line where $x_1 = \dfrac{d_1 c}{d_1^2 + d_2^2}$ and $x_2 = \dfrac{d_2 c}{d_1^2 + d_2^2}$, since

$$d_1 x_1 + d_2 x_2 = d_1\left(\frac{d_1 c}{d_1^2+d_2^2}\right) + d_2\left(\frac{d_2 c}{d_1^2+d_2^2}\right)$$

$$= \frac{d_1^2 c + d_2^2 c}{d_1^2 + d_2^2} = c\left(\frac{d_1^2+d_2^2}{d_1^2+d_2^2}\right) = c.$$

With a vector representation of the line, we take the translation vector as

$\vec{a} = \left[\dfrac{d_1 c}{d_1^2+d_2^2}, \dfrac{d_2 c}{d_1^2+d_2^2}\right]$. The line is then

$\vec{x} = t\vec{d} + \vec{a}$ and so parametrically we have

$$\boxed{\begin{aligned} x_1 &= t d_2 + \frac{d_1 c}{d_1^2+d_2^2} \\ x_2 &= -t d_1 + \frac{d_2 c}{d_1^2+d_2^2} \end{aligned}}$$

(notice this is valid for either $d_1 = 0$ or $d_2 = 0$).

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For $d_2 \neq 0$, if we let $t = \dfrac{s}{d_2} - \dfrac{d_1 c}{(d_1^2+d_2^2)\,d_2}$

then $x_1 = t\,d_2 + \dfrac{d_1 c}{d_1^2+d_2^2} = \left(\dfrac{s}{d_2} - \dfrac{d_1 c}{(d_1^2+d_2^2)\,d_2}\right)d_2 + \dfrac{d_1 c}{d_1^2+d_2^2}$

$$= s - \frac{d_1 c}{d_1^2+d_2^2} + \frac{d_1 c}{d_1^2+d_2^2} = s$$

and $x_2 = -t\,d_1 + \dfrac{d_2 c}{d_1^2+d_2^2} = -\left(\dfrac{s}{d_2} - \dfrac{d_1 c}{(d_1^2+d_2^2)\,d_2}\right)d_1 + \dfrac{d_2 c}{d_1^2+d_2^2}$

$$= -\left(\frac{d_1}{d_2}\right)s + \frac{d_1^2 c}{(d_1^2+d_2^2)\,d_2} + \frac{d_2^2 c}{(d_1^2+d_2^2)\,d_2}$$

$$= -\left(\frac{d_1}{d_2}\right)s + \frac{(d_1^2+d_2^2)\,c}{(d_1^2+d_2)^2\,d_2} = -\left(\frac{d_1}{d_2}\right)s + \frac{c}{d_2}.$$

So for $d_2 \neq 0$, our solution agrees with the text's solution (by properly adjusting the parameter). Similarly, if $d_2 = 0$ we can let

$t = \dfrac{-s}{d_1} + \dfrac{d_2 c}{(d_1^2+d_2^2)\,d_1}$ and we find

$x_1 = t\,d_2 + \dfrac{d_1 c}{d_1^2+d_2^2} = \left(\dfrac{-s}{d_1} + \dfrac{d_2 c}{(d_1^2+d_2^2)\,d_1}\right)d_2 + \dfrac{d_1 c}{d_1^2+d_2^2}$

$= -\left(\dfrac{d_2}{d_1}\right)s + \dfrac{c}{d_1}$ and

$x_2 = -t\,d_1 + \dfrac{d_2 c}{d_1^2+d_2^2} = -\left(\dfrac{-s}{d_1} + \dfrac{d_2 c}{(d_1^2+d_2^2)\,d_1}\right)d_1 + \dfrac{d_2 c}{d_1^2+d_2^2}$

$= s$,

and for $d_2 = 0$ our solution agrees with the text's solution (by properly adjusting the parameter). $\square$