3.2.21 Determine whether or not the set of vectors $\{x, (x+1)^2, (x-1)^2\}$ is a basis for the vector space P_2 of all polynomials of degree 2 or less.

Solution

We use Definition 3.6, "Basis for a Vector Space", to see if the set is a linearly independent spanning set. For linear independence we consider the equation $(r_1)x + r_2(x+1)^2 + r_3(x-1)^2 = 0x^2+0x+0 = 0$. This gives

$$r_1 x + r_2(x^2+2x+1) + r_3(x^2-2x+1) = 0$$

or $(r_2+r_3)x^2 + (r_1+2r_2-2r_3)x + (r_2+r_3) = 0.$

This gives the three equations in unknowns r_1, r_2, r_3:

$$\begin{aligned} r_2+r_3&=0 \\ r_1+2r_2-2r_3&=0 \\ r_2+r_3&=0 \end{aligned}$$

and we consider $\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 2 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

We $\sim$ $\left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ which gives

$$\begin{aligned} r_1 - 4r_3 &= 0 \\ r_2+r_3 &= 0 \\ 0 &= 0. \end{aligned}$$

So there are multiple values of r_1, r_2, r_3 that satisfy the equations. In particular, we CAN take $r_1 = 4$, $r_2 = -1$, $r_3 = 1$. This gives the dependence relation

$$4(x) - 1(x+1)^2 + 1(x-1)^2 = 0.$$

So the vectors are not linearly independent and hence the set of vectors

$\boxed{\text{is not a basis for } P_2.}$ $\square$