SECTION 3.2

EXERCISE #31

3.2.31 Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a basis for a vector space V. Prove that the vectors $\vec{w}_1 = \vec{v}_1 + \vec{v}_2$, $\vec{w}_2 = \vec{v}_2 + \vec{v}_3$, $\vec{w}_3 = \vec{v}_1 - \vec{v}_3$ do not generate V.

Proof

Notice that $\vec{w}_1 - \vec{w}_2 = (\vec{v} + \vec{v}_2) - (\vec{v}_2 + \vec{v}_3) = \vec{v}_1 - \vec{v}_3 = \vec{w}_3$, so $\vec{w}_3 \in sp(\vec{w}_1, \vec{w}_2)$. [So we expect the dimension of $sp(\vec{w}_1, \vec{w}_2)$ to be 2 whereas the dimension of V is 3; to complete the proof we need to find an element of V that is not in $sp(\vec{w}_1; \vec{w}_2, \vec{w}_3)$.] Now $\vec{v}_1 \in V$. ASSUME that $\vec{v}_1 \in sp(\vec{w}_1, \vec{w}_2, \vec{w}_3)$. Then

$$\begin{aligned} \vec{v}_1 &= r_1 \vec{w}_1 + r_2 \vec{w}_2 + r_3 \vec{w}_3 \quad \text{for some } r_1, r_2, r_3 \in \mathbb{R} \\ &= r_1(\vec{v}_1 + \vec{v}_2) + r_2(\vec{v}_2 + \vec{v}_3) + r_3(\vec{v}_1 - \vec{v}_3) \\ &= (r_1 + r_3)\vec{v}_1 + (r_1 + r_2)\vec{v}_2 + (r_2 - r_3)\vec{v}_3, \end{aligned}$$

or

$$(r_1 + r_3 - 1)\vec{v}_1 + (r_1 + r_2)\vec{v}_2 + (r_2 - r_3)\vec{v}_3 = \vec{0}.$$

Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for V then by Definition 3.6, "Basis for a Vector Space," then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent and so we must have

$$\begin{aligned} r_1 + r_3 &= 1 \\ r_1 + r_2 &= 0 \\ r_2 - r_3 &= 0 \end{aligned}$$

, so there must be a solution to this system of equations. Consider the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right],$$

but this implies that $0 = 1$ and so there is no solution to the system of equations, a CONTRADICTION. So the assumption that $\vec{v}_1 \in sp(\vec{w}_1, \vec{w}_2, \vec{w}_3)$ is false and so $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is not a spanning set of V and hence (by Definition 3.6) is not a basis for V. ■