SECTION 3.2

EXERCISE #33

3.2.33 Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for a vector space V and let $\vec{w} = t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k$ with $t_k \neq 0$. Prove that $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k-1}, \vec{w}, \vec{v}_{k+1}, \ldots, \vec{v}_n\}$ is a basis for V.

<u>Proof</u>

First, since $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis then by Definition 3.6, "Basis for a Vector Space," it is a spanning set for V. Let $\vec{v} \in V$. Then $\vec{v} = r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_n\vec{v}_n$ for some $r_1, r_2, \ldots, r_n \in \mathbb{R}$.

$$= r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_{k-1}\vec{v}_{k-1} + r_k\vec{v}_k + r_{k+1}\vec{v}_{k+1} + \cdots + r_n\vec{v}_n$$

$$= r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_{k-1}\vec{v}_{k-1} + r_k\left(\frac{\vec{w} - t_1\vec{v}_1 - t_2\vec{v}_2 - \cdots - t_{k-1}\vec{v}_{k-1}}{t_k}\right) + r_{k+1}\vec{v}_{k+1} + \cdots + r_n\vec{v}_n \quad \text{since } t_k \neq 0$$

$$= \left(r_1 - \frac{t_1}{t_k}\right)\vec{v}_1 + \left(r_2 - \frac{t_2}{t_k}\right)\vec{v}_2 + \cdots + \left(r_{k-1} - \frac{t_{k-1}}{t_k}\right)\vec{v}_{k-1} + \frac{r_k}{t_k}\vec{w} + r_{k+1}\vec{v}_{k+1} + \cdots + r_n\vec{v}_n,$$

so $\vec{v} \in sp(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k-1}, \vec{w}, \vec{v}_{k+1}, \ldots, \vec{v}_n)$ and since $\vec{v}$ is an arbitrary vector in V then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k-1}, \vec{w}, \vec{v}_{k+1}, \ldots, \vec{v}_n\}$ is a spanning set.

Second, we must show linear independence of $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k+1}, \vec{w}, \vec{v}_{k+1}, \ldots, \vec{v}_n\}$. Suppose

$$r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_{k-1}\vec{v}_{k+1} + r_k\vec{w} + r_{k+1}\vec{v}_{k+1} + \cdots + r_n\vec{v}_n = \vec{0}.$$

Then $r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_{k-1}\vec{v}_{k-1} + r_k(t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k) + r_{k+1}\vec{v}_{k+1} + \cdots + r_n\vec{v}_n = \vec{0}$, or

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EXERCISE #33 (continued)

$$(r_1 + r_k t_1)\vec{v}_1 + (r_2 + r_k t_2)\vec{v}_2 + \cdots + (r_{k-1} + r_k t_k)\vec{v}_{k-1}$$
$$+ r_k t_k \vec{v}_k + r_{k+1}\vec{v}_{k+1} + \cdots + r_n \vec{v}_n = \vec{0}$$

Since, being a basis, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent (Definition 3.6) and so $r_1 + r_k t_1 = r_2 + r_k t_2 = \cdots = r_{k-1} + r_k t_k = r_k t_k = r_{k+1} = \cdots = r_n = 0$.
So $r_{k+1} = r_{k+2} = \cdots = r_n = 0$ and, since $t_k \neq 0$, $r_k = 0$. So $r_1 + r_k t_1 = r_1 = 0$, $r_2 + r_k t_2 = r_2 = 0$, $\ldots$, $r_{k-1} + r_k t_k = r_{k-1} = 0$. That is, $r_1 = r_2 = \cdots = r_n = 0$.
So $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k-1}, \vec{w}, \vec{v}_{k+1}, \ldots, \vec{v}_n\}$ is linearly independent.

Since $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k-1}, \vec{w}, \vec{v}_{k+1}, \ldots, \vec{v}_n\}$ is a linearly independent spanning set in V then by Definition 3.6 it is a basis for V. ■