SECTION 3.2
EXERCISE #37

3.2.37 Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a list of nonzero vectors in a vector space V such that no vector in this list is a linear combination of its predecessors. Prove that the vectors in the list form an independent set.

Proof

ASSUME $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a dependent set. Then $r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_n\vec{v}_n = \vec{0}$ where not all $r_i = 0$.
Let k be the largest index such that $r_k \neq 0$.
Then $r_{k+1} = r_{k+2} = \cdots = r_n$ and

$r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_{k-1}\vec{v}_{k-1} + r_k\vec{v}_k = \vec{0}$ or $-r_k\vec{v}_k = r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_{k-1}\vec{v}_{k-1}$.

or (since $r_k \neq 0$) $\vec{v}_k = \left(\dfrac{-r_1}{r_k}\right)\vec{v}_1 + \left(\dfrac{-r_2}{r_k}\right)\vec{v}_2 + \cdots + \left(\dfrac{-r_{k-1}}{r_k}\right)\vec{v}_{k-1}$.

But then $\vec{v}_k$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k-1}$, CONTRADICTING the hypothesis that no vector is a linear combination of its predecessors. So the assumption that $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is dependent is false and, in fact, the set is independent. ■