

SECTION 3.2
EXERCISE #39

3.2.39 We once watched a speaker in a lecture derive the equation $f(x)\sin(x) + g(x)\cos(x) = 0$ and then say, "Now everyone knows that $\sin(x)$ and $\cos(x)$ are independent functions, so $f(x) = 0$ and $g(x) = 0$." Was the statement correct or incorrect? Give a proof or a counterexample.

Solution

To say that $\sin(x)$ and $\cos(x)$ are independent means that we are treating them as vectors and that the linear combination $r_1\sin(x) + r_2\cos(x) = 0$ for scalars r_1 and r_2 (not functions; and the equation is understood to hold for all x in some given set). So it is not appropriate to use the term "independent" in the setting of $f(x)\sin(x) + g(x)\cos(x) = 0$ since this is not a linear combination of $\sin(x)$ and $\cos(x)$!

In addition, we can violate the claim that $f(x)$ and $g(x)$ must be the 0 function.

We can take $f(x) = \cos(x)$ and $g(x) = -\sin(x)$ to get $f(x)\sin(x) + g(x)\cos(x) = \cos(x)\sin(x) + (-\sin(x))\cos(x) = 0$. \square