SECTION 3.2

EXERCISE #41

3.2.41 Referring to Exercise #40, suppose that the differential equation

$$f_n(x)\,y^{(n)} + f_{n-1}(x)\,y^{(n-1)} + \dots + f_2(x)\,y'' + f_1(x)\,y' + f_0(x)\,y = g(x)$$

does have a solution $y = p(x)$ in the space F of all functions mapping $\mathbb{R}$ into $\mathbb{R}$. By analogy with Theorem 1.18, "Structure of the Solution Set of $A\vec{x} = \vec{b}$" (see Section 1.6, "Homogeneous Systems, Subspaces, and Bases"), describe the structure of the set of solutions of this equation that lie in F.

Solution

In Theorem 1.18 we see that every solution of $A\vec{x} = \vec{b}$ is of the form $\vec{x} = \vec{p} + \vec{h}$ where $\vec{p}$ is a particular solution so that $A\vec{p} = \vec{b}$, and $\vec{h}$ is any solution to the associated homogeneous system $A\vec{x} = \vec{0}$. In Exercise #40 we see that solutions to the associated homogeneous differential equation

$$f_n(x)\,y^{(n)} + \dots + f_2(x)\,y'' + f_1(x)\,y' + f_0(x)\,y = 0$$

form a subspace of F; denote this subspace as H. Since $p(x)$ is a particular solution of the original differential equation, we expect every solution of the original differential equation to be of the form $p(x) + h(x)$ where $h(x) \in H$. $\square$