SECTION 3.2
EXERCISE # 43

3.2.43 Use Exercise # 41 to describe the solution set of the given differential equation:

(a) $y'' + y = 0$.

Solution

We need $y'' = -y$. That is, we need functions that when we differentiate them twice we get the negative of the original functions. Well $h_1(x) = \cos(x)$ and $h_2(x) = \sin(x)$ both satisfy this property since

$$\frac{d^2}{dx^2}[\cos(x)] = \frac{d}{dx}[-\sin(x)] = -\cos(x) \text{ and}$$

$$\frac{d^2}{dx^2}[\sin(x)] = \frac{d}{dx}[\cos(x)] = -\sin(x).$$

Since the differential equation is second order (it involves second derivatives) then we expect two independent solutions (see the comment in the text after Exercise # 42). So the solution space is

$$\boxed{H = \{r_1 \cos(x) + r_2 \sin(x) \mid r_1, r_2 \in \mathbb{R}\}.} \quad \square$$

(b) $y'' + y = x$.

Solution

By inspection, we notice that $p(x) = x$ is a particular solution. So by Exercise # 41, every solution is of the form $p(x) + h(x)$ where $h(x) \in H$. That is, the general solution is

$$\boxed{x + r_1 \cos(x) + r_2 \sin(x) \text{ where } r_1, r_2 \in \mathbb{R}.} \quad \square$$