SECTION 3.3
EXERCISE # 19

3.3.19 Prove that $\{1, \sin(x), \cos(x), \sin(2x), \cos(2x)\}$ is an independent set of functions in the vector space F of all functions mapping $\mathbb{R}$ into $\mathbb{R}$.

Proof

Consider $r_1(1) + r_2\sin(x) + r_3\cos(x) + r_4\sin(2x) + r_4\cos(2x) = 0$ (which must hold for all $x \in \mathbb{R}$). We cleverly choose 5 values of x to produce 5 equations in the unknown coefficients (this requires some insight; of course choosing $x = 0$ and $x = 2\pi$ will produce the same equation).

With $x = 0$ we have

$$r_1 + r_2\sin(0) + r_3\cos(0) + r_4\sin(2(0)) + r_4\cos(2(0)) = 0$$

or $r_1 + r_3 + r_4 = 0$. With $x = \pi/4$ we have

$$r_1 + r_2\sin\left(\frac{\pi}{4}\right) + r_3\cos\left(\frac{\pi}{4}\right) + r_4\sin\left(2\left(\frac{\pi}{4}\right)\right) + r_5\cos\left(2\left(\frac{\pi}{4}\right)\right) = 0$$

or $r_1 + \frac{\sqrt{2}}{2}r_2 + \frac{\sqrt{2}}{2}r_3 + r_4 = 0$. With $x = -\pi/4$ we have

$$r_1 + r_2\sin\left(\frac{-\pi}{4}\right) + r_3\cos\left(\frac{-\pi}{4}\right) + r_4\sin\left(2\left(\frac{-\pi}{4}\right)\right) + r_5\cos\left(2\left(\frac{-\pi}{4}\right)\right) = 0$$

or $r_1 - \frac{\sqrt{2}}{2}r_2 + \frac{\sqrt{2}}{2}r_3 - r_4 = 0$. With $x = \frac{\pi}{2}$ we have

$$r_1 + r_2\sin\left(\frac{\pi}{2}\right) + r_3\cos\left(\frac{\pi}{2}\right) + r_4\sin\left(2\left(\frac{\pi}{2}\right)\right) + r_5\cos\left(2\left(\frac{\pi}{2}\right)\right) = 0$$

or $r_1 + r_2 - r_5 = 0$. With $x = \frac{-\pi}{2}$ we have

$$r_1 + r_2\sin\left(\frac{-\pi}{2}\right) + r_3\cos\left(\frac{-\pi}{2}\right) + r_4\sin\left(2\left(\frac{-\pi}{2}\right)\right) + r_5\cos\left(2\left(\frac{-\pi}{2}\right)\right) = 0$$

or $r_1 - r_2 - r_5 = 0$. So we need

$$\begin{aligned}
r_1 \qquad\quad + r_3 + r_4 \quad &= 0 \\
r_1 + \tfrac{\sqrt{2}}{2}r_2 + \tfrac{\sqrt{2}}{2}r_3 + r_4 \quad &= 0 \\
r_1 - \tfrac{\sqrt{2}}{2}r_2 + \tfrac{\sqrt{2}}{2}r_3 - r_4 \quad &= 0 \\
r_1 + r_2 \qquad\qquad - r_5 &= 0 \\
r_1 - r_2 \qquad\qquad - r_5 &= 0.
\end{aligned}$$

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So we consider the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & \sqrt{2}/2 & \sqrt{2}/2 & 1 & 0 & 0 \\ 1 & -\sqrt{2}/2 & \sqrt{2}/2 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \end{array}\right] \xrightarrow[R_5 \to R_5 - R_4]{R_3 \to R_3 - R_2} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & \sqrt{2}/2 & \sqrt{2}/2 & 1 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{R_3 \to R_3 - \frac{\sqrt{2}}{2} R_5 \\ R_4 \to R_3 + \frac{1}{2} R_5}]{R_2 \to R_2 + \frac{\sqrt{2}}{4} R_5} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & \sqrt{2}/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\substack{R_4 \to -R_4 \\ R_5 \to R_5/(-2)}]{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3/(-2)}} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2}-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2/(\frac{\sqrt{2}}{2}-1)]{R_1 \to R_1 - R_3} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

and so we must have $r_1 = r_2 = r_3 = r_4 = r_5 = 0$. So by Definition 3.5, "Linear Dependence and Independence", the set is linearly independent. □

(b) Find a basis for the subspace of F generated by the functions

$f_1(x) = 1 - 2\sin(x) + 4\cos(x) - \sin(2x) - 3\cos(2x)$
$f_2(x) = 2 - 3\sin(x) - \cos(x) + 4\sin(2x) + 5\cos(2x)$
$f_3(x) = 5 - 8\sin(x) + 2\cos(x) + 7\sin(2x) + 7\cos(2x)$
$f_4(x) = -1 + 14\cos(x) - 11\sin(2x) - 19\cos(2x)$

<u>Solution</u>

We introduce the ordered basis

$B = (1, \sin(x), \cos(x), \sin(2x), \cos(2x))$ so that

$f_1(x)_B = [1, -2, 4, -1, -3]$ and
$f_2(x)_B = [2, -3, -1, 4, 5]$ $f_4(x)_B = [-1, 0, 14, -11, -19]$.
$f_3(x)_B = [5, -8, 2, 7, 7]$

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We now apply Theorem 2.1.A, "Finding a Basis for $W = sp(\vec{w}_1, \vec{w}_2, \dots, \vec{w}_k)$," to the coordinate vectors:

$$A = \begin{bmatrix} 1 & 2 & 5 & -1 \\ -2 & -3 & -8 & 0 \\ 4 & -1 & 2 & 14 \\ -1 & 4 & 7 & -11 \\ -3 & 5 & 7 & -19 \end{bmatrix} \xrightarrow[\substack{R_4 \to R_4 + R_1 \\ R_5 \to R_5 + 3R_1}]{\substack{R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 4R_1}} \begin{bmatrix} 1 & 2 & 5 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & -9 & -18 & 18 \\ 0 & 6 & 12 & -12 \\ 0 & 11 & 22 & -22 \end{bmatrix}$$

$$\xrightarrow[\substack{R_4 \to R_4/6 \\ R_5 \to R_5/11}]{R_3 \to R_3/(-9)} \begin{bmatrix} 1 & 2 & 5 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 1 & 2 & -2 \\ 0 & 1 & 2 & -2 \\ 0 & 1 & 2 & -2 \end{bmatrix} \xrightarrow[\substack{R_4 \to R_4 - R_2 \\ R_5 \to R_5 - R_2}]{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 5 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = H.$$

Now H is in row echelon form and columns 1 and 2 contain pivots so by Theorem 2.1.A a basis for
$sp([1,-2,4,-1,-3],[2,-3,-1,4,5],[5,-8,2,7,7],[-1,0,14,-11,-19])$
is $[1,-2,4,-1,-3] = f_1(x)_B$ and $[2,-3,-1,4,5] = f_2(x)_B$.
So a basis for $sp(f_1(x), f_2(x), f_3(x), f_4(x))$
is

$$\boxed{\begin{aligned} \{ f_1(x) &= 1 - 2\sin(x) + 4\cos(x) - \sin(2x) - 3\cos(2x), \\ f_2(x) &= 2 - 3\sin(x) - \cos(x) + 4\sin(2x) + 5\cos(2x) \}. \end{aligned}}$$

□