SECTION 3.4
NUMBER 41

3.4.41 Exercise 40 states: "Let V and V' be vector spaces, let $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a basis for V, and let $\vec{c}_1{}', \vec{c}_2{}', \ldots, \vec{c}_n{}' \in V'$. Prove that there exists a linear transformation $T: V \to V'$ such that $T(\vec{b}_i) = \vec{c}_i{}'$ for $i = 1, 2, \ldots, n$. State and prove a generalization of Exercise 40 for any vector spaces V and V', where V has a basis B.

Solution

Since the dimensions of V and V' would be different (one could even be finite while the other is infinite). So we with a basis B of V and consider any corresponding set C' of elements in V'. Then we create a linear transformation which maps each basis element to the corresponding element of C'. We let the "correspondence" be represented by any function $f: B \to V'$. (We could take this in other directions, especially if we make assumptions about the dimensions of V and V'. But this approach offers us the most general approach.) We take as the generalization:

"Let V and V' be vector spaces, let B be a basis for V, and let f be any function mapping B into V'. Denote $\vec{c}_\alpha{}' = f(\vec{b}_\alpha)$ for each $\vec{b}_\alpha \in B$ (we take $\alpha \in A$ where A is any indexing set for basis B; in Exercise 40

SECTION 3.4

NUMBER 41 (continued)

the indexing set is $A = \{1, 2, \ldots, n\}$. Then there is a linear transformation $T: V \to V'$ such that $T(\vec{b}_\alpha) = \vec{c}\,'_\alpha$ for $\alpha \in A$."

Proof

For any $\vec{v} \in V$, since B is a basis for V, there are $\vec{b}_{\alpha_1}, \vec{b}_{\alpha_2}, \ldots, \vec{b}_{\alpha_n} \in B$ and unique scalars $r_1, r_2, \ldots, r_n \in \mathbb{R}$ such that $r_1\vec{b}_{\alpha_1} + r_2\vec{b}_{\alpha_2} + \cdots + r_n\vec{b}_{\alpha_n} = \vec{v}$. For such unique scalars, define

$$T(\vec{v}) = r_1\vec{c}\,'_{\alpha_1} + r_2\vec{c}\,'_{\alpha_2} + \cdots + r_n\vec{c}\,'_{\alpha_n}.$$

Notice that this definition gives $T(\vec{b}_\alpha) = \vec{c}\,'_\alpha$ for all $\alpha \in A$, as desired.

Now we show that T is linear. Let $\vec{v}, \vec{w} \in V$ and $r, s \in \mathbb{R}$. Let $\vec{v}$ be represented as given above. Similarly for $\vec{w}$, there are $\vec{b}_{\alpha_1^*}, \vec{b}_{\alpha_2^*}, \ldots, \vec{b}_{\alpha_m^*} \in B$ and unique scalars $s_1, s_2, \ldots, s_n \in \mathbb{R}$ such that $s_1\vec{b}_{\alpha_1^*} + s_2\vec{b}_{\alpha_2^*} + \cdots + s_m\vec{b}_{\alpha_m^*} = \vec{w}$, so that $T(\vec{w}) = s_1\vec{c}\,'_{\alpha_1^*} + s_2\vec{c}\,'_{\alpha_2} + \cdots + s_m\vec{c}\,'_{\alpha_m^*}$.

We change notation a little. Some of the $\vec{b}_{\alpha_i}$ and $\vec{b}_{\alpha_j^*}$ may be the same. Consider the set $\{\vec{b}_{\alpha_1}, \vec{b}_{\alpha_2}, \ldots, \vec{b}_{\alpha_n}, \vec{b}_{\alpha_1^*}, \vec{b}_{\alpha_2^*}, \ldots, \vec{b}_{\alpha_m^*}\}$ and let the size of this set be k (so that $k \le m + n$). Denote vectors in this set as $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k$ and let $\vec{v} = p_1\vec{b}_1 + p_2\vec{b}_2 + \cdots + p_k\vec{b}_k$ and

$\vec{w} = q_1 \vec{b}_1 + q_2 \vec{b}_2 + \cdots + q_k \vec{b}_k$ (so that some of the p_i's equal some of the r_j's, some of the q_i's equal some of the s_j's; we expect many of the p's and q's to be 0). Notice that the definition of T gives

$T(\vec{v}) = p_1 \vec{c}_1' + p_2 \vec{c}_2' + \cdots + p_k \vec{c}_k'$ and

$T(\vec{w}) = q_1 \vec{c}_1' + p_2 \vec{c}_2' + \cdots + q_k \vec{c}_k'$ where

$\vec{c}_i' = T(\vec{b}_i)$ for $i = 1, 2, \ldots, k$. Next,

$$T(r\vec{v} + s\vec{w}) = T(r(p_1 \vec{b}_1 + p_2 \vec{b}_2 + \cdots + p_k \vec{b}_k))$$
$$+ T(s(q_1 \vec{b}_1 + q_2 \vec{b}_2 + \cdots + q_k \vec{b}_k))$$
$$= T((rp_1)\vec{b}_1 + (rp_2)\vec{b}_2 + \cdots + (rp_k)\vec{b}_k)$$
$$+ T((sq_1)\vec{b}_1 + (sq_2)\vec{b}_2 + \cdots + (sq_k)\vec{b}_k)$$
$$= (rp_1 \vec{c}_1' + rp_2 \vec{c}_2' + \cdots + rp_k \vec{c}_k')$$
$$+ (sq_1 \vec{c}_1' + sq_2 \vec{c}_2' + \cdots + sq_k \vec{c}_k')$$ by the definition of T

$$= r(p_1 \vec{c}_1' + p_2 \vec{c}_2' + \cdots + p_k \vec{c}_k') + s(q_1 \vec{c}_1' + q_2 \vec{c}_2' + \cdots + q_k \vec{c}_k')$$
$$= rT(\vec{v}) + sT(\vec{w}).$$

Therefore, T is linear as required. ∎