SECTION 3.4

NUMBER 47



3.4.47 Prove that for any five linear transformations $T_1, T_2, T_3, T_4, T_5$ mapping $\mathbb{R}^2$ into $\mathbb{R}^2$, there exists scalars $c_1, c_2, c_3, c_4, c_5$ (not all of which are zero) such that $T = c_1T_1 + c_2T_2 + c_3T_3 + c_4T_4 + c_5T_5$ has the property that $T(\vec{x}) = \vec{0}$ for all $\vec{x} \in \mathbb{R}^2$.

Proof

By the corollary of Section 2.3, "Standard Matrix Representation of a Linear Transformation", there are $2\times 2$ standard matrix representations $A_1, A_2, A_3, A_4, A_5$ of $T_1, T_2, T_3, T_4, T_5$, respectively. Now the set of all $2\times 2$ matrices form a vector space (see Example 1 in Section 3.1) of dimension 4 (see Illustration 11 in Section 3.2). By Theorem 3.4, "Relative Size of Spanning and Independent Sets," a set of 5 vectors in vector space of dimension 4 must be dependent. So by Definition 3.5, "Linear Dependence and Independence," there are scalars $c_1, c_2, c_3, c_4, c_5$, not all zero, such that $c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 + c_5A_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Then for any $\vec{x} \in \mathbb{R}^2$ we have

$$(c_1T_1 + c_2T_2 + c_3T_3 + c_4T_4 + c_5T_5)(\vec{x})$$
$$= (c_1T_1)(\vec{x}) + (c_2T_2)(\vec{x}) + (c_3T_3)(\vec{x}) + (c_4T_4)(\vec{x}) + (c_5T_5)(\vec{x})$$

by the definition of addition of linear transformations

SECTION 3.4

NUMBER 47 (continued)



$= c_1 \; T_1(\vec{x}) + c_2 \; T_2(\vec{x}) + c_3 \; T_3(\vec{x}) + c_4 \; T_4(\vec{x}) + c_5 \; T_5(\vec{x})$

by the definition of scalar multiplication of linear transformations

$= c_1 A_1 \vec{x} + c_2 A_2 \vec{x} + c_3 A_3 \vec{x} + c_4 A_4 \vec{x} + c_5 A_5 \vec{x}$

$= (c_1 A_1 + c_2 A_2 + c_3 A_3 + c_4 A_4 + c_5 A_5) \vec{x}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}.$

That is, $T \vec{x} = (c_1 T_1 + c_2 T_2 + c_3 T_3 + c_4 T_4 + c_5 T_5) \vec{x} = \vec{0}$ for all $\vec{x} \in \mathbb{R}^2$, as claimed. ∎