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3.4.49 Let V and V' be vector spaces having the same finite dimension, and let $T: V \to V'$ be a linear transformation. Prove that T is one-to-one if and only if range$(T) = V'$.

HINT: Use Exercise 36 in Section 3.2 which states: "If W is a subspace of an n-dimensional vector space V and $\dim(W) = n$, then $W = V$."

<u>Proof</u>

Let $\dim(V) = \dim(V') = n$.

Suppose T is one-to-one. Then $T: V \to$ range(T) is a one-to-one and onto linear transformation from V to range(T). That is, T is an isomorphism of V with range(T) (see Section 3.3). So range(T) is an n-dimensional subspace of n-dimensional space V'. By Exercise 36 in Section 3.2, range$(T) = V'$.

Next, suppose T is not one-to-one. Then by Corollary 3.4.A, "One-to-One and Kernel," $\ker(T) \neq \{\vec{0}\}$. Now $\ker(T) = T^{-1}(\{\vec{0}\})$ by definition and so $\ker(T)$ is a subspace of V by Theorem 3.7(2), "Preservation of Subspaces." So let $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k\}$ be a basis for $\ker(T)$ (since $\ker(T) \neq \{\vec{0}\}$ then $k \geq 1$). By Theorem 2.3(2), "Existence and Determination of Bases," $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k\}$ can be extended to a basis of V, say $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$.

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(Notice that Theorem 2.3 addresses vector space $\mathbb{R}^n$ and subspaces of $\mathbb{R}^n$, but we know by the Fundamental Theorem of Finite Dimensional Vector Spaces, Theorem 3.3.A, that V' is isomorphic to $\mathbb{R}^n$ and so the result also holds for V.) So

$$\text{range}(V) = \{T(\vec{v}) \mid \vec{v} \in V\} = \{T(\vec{v}) \mid \vec{v} \in sp(\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)\}$$
$$= sp(T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_k), T(\vec{b}_{k+1}), \ldots, T(\vec{b}_n))$$
$$= sp(T(\vec{b}_{k+1}), T(\vec{b}_{k+2}), \ldots, T(\vec{b}_n))$$
$$\text{since } T(\vec{b}_1) = T(\vec{b}_2) = \cdots = T(\vec{b}_k) = \vec{0}' \in V'.$$

But then $\dim(\text{range}(T)) \le n - k < n$. So $\text{range}(T) \ne V'$ (since $\dim(V') = n$). We have shown that if T is not one-to-one then $\text{range}(T) \ne V'$. This is logically equivalent to the statement (the contrapositive of what has been shown): "If $\text{range}(T) = V'$ then T is one-to-one." ∎