SECTION 3.4

NUMBER 51

3.4.51 Give an example of a vector space V and a linear transformation $T: V \to V$ such that range$(T) = V$ but T is not one-to-one.

Solution

Consider the vector space $V = \mathbb{P}$ of all polynomials (see Example 2 of Section 3.1). Let $T: V \to V$ be differentiation so that $T(p(x)) = p'(x)$. We know T is linear from our knowledge of calculus since:

$$T(rp(x) + sq(x)) = \frac{d}{dx}[rp(x) + sq(x)] = rp'(x) + sq'(x)$$
$$= rT(p(x)) + sT(q(x)).$$

For any $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in V$ we have

$$T(a_0 x + a_1 x^2/2 + a_2 x^3/3 + \cdots + a_n x^{n+1}/(n+1))$$
$$= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

so range$(T) = V$. However, T is not one-to-one since, for example,

$$T(x^2 + 2) = \frac{d}{dx}[x^2 + 2] = 2x = \frac{d}{dx}[x^2] = T(x^2).$$

$\square$