SECTION 3.5

NUMBER 17

3.5.17 For $\vec{v}$ and $\vec{w}$ in an inner-product space, prove that $\vec{v}-\vec{w}$ and $\vec{v}+\vec{w}$ are perpendicular if and only if $\|\vec{v}\| = \|\vec{w}\|$.

Proof

Notice that

$\langle \vec{v}-\vec{w}, \vec{v}+\vec{w} \rangle = \langle \vec{v},\vec{v}\rangle - \langle \vec{w},\vec{v}\rangle + \langle \vec{v},\vec{w}\rangle - \langle \vec{w},\vec{w}\rangle$ by the properties of an inner product, Definition 3.12

$= \langle \vec{v},\vec{v}\rangle - \langle \vec{w},\vec{w}\rangle$ since $\langle \vec{v},\vec{w}\rangle - \langle \vec{w},\vec{v}\rangle = \langle \vec{v},\vec{w}\rangle - \langle \vec{v},\vec{w}\rangle = 0$ (Definition 3.12(P1))

$= \|\vec{v}\|^2 - \|\vec{w}\|^2$.

So $\vec{v}-\vec{w}$ and $\vec{v}+\vec{w}$ are perpendicular if and only if $\langle \vec{v}-\vec{w}, \vec{v}+\vec{w}\rangle = 0$ which, in turn, holds if and only if $\|\vec{v}\|^2 = \|\vec{w}\|^2$ which holds if and only if $\|\vec{v}\| = \|\vec{w}\|$, as claimed. ■