SECTION 3.5

NUMBER 19

3.5.19 Let $S$ be a subset of nonzero vectors in an inner-product space $V$, and suppose that any two different vectors in $S$ are orthogonal. Prove that $S$ is an independent set.

<u>Proof</u>

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in S$ and suppose $r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_n\vec{v}_n = \vec{0}$ for scalars $r_1, r_2, \ldots, r_n \in \mathbb{R}$. Let $j \in \{1, 2, \ldots, n\}$. Then $\vec{v}_j \in S$ and

$$0 = \langle \vec{v}_j, \vec{0} \rangle = \langle \vec{v}_j, r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_n\vec{v}_n \rangle$$

$$= r_1 \langle \vec{v}_j, \vec{v}_1 \rangle + r_2 \langle \vec{v}_j, \vec{v}_2 \rangle + \cdots + r_j \langle \vec{v}_j, \vec{v}_j \rangle + \cdots + r_n \langle \vec{v}_j, \vec{v}_n \rangle$$ by the Properties of an Inner Product as given in Definition 3.12

$$= 0 + 0 + \cdots + 0 + r_j \langle \vec{v}_j, \vec{v}_j \rangle + 0 + \cdots + 0$$ since $\langle \vec{v}_j, \vec{v}_i \rangle = 0$ for $i \neq j$ by the definition of set $S$

$$= r_j \langle \vec{v}_j, \vec{v}_j \rangle = r_j \|\vec{v}_j\|^2.$$

Since $\|\vec{v}_j\| \neq 0$ (because $\vec{v}_j \neq \vec{0}$ by the definition of $S$) then it must be that $r_j = 0$. Since this holds for all $j \in \{1, 2, \ldots, n\}$ then $r_1 = r_2 = \cdots = r_n = 0$ and so, by Definition 3.5, "Linear Dependence and Independence," set $S$ consists of linearly independent vectors. ∎