SECTION 3.5
NUMBER 27

3.5.27 Consider the space $C_{a,b}$ of continuous functions with domain the closed interval $a \le x \le b$, and let $w(x)$ be a positive continuous weight function, so that $w(x) > 0$ for $a \le x \le b$. Prove that for f and g in $C_{a,b}$, the weighted integral

$$\langle f, g \rangle = \int_a^b w(x) f(x) g(x)\, dx$$

defines an inner product on $C_{a,b}$.

Proof

We must verify the 4 properties of an inner product given in Definition 3.12. Let $f, g, h \in C_{a,b}$ and let $r \in \mathbb{R}$.

P1 $\langle f, g \rangle = \int_a^b w(x) f(x) g(x)\, dx$

$$= \int_a^b w(x) g(x) f(x)\, dx = \langle g, f \rangle.$$

P2 $\langle f, g+h \rangle = \int_a^b w(x) f(x) \left((g+h)(x) \right) dx$

$$= \int_a^b w(x) f(x) \left(g(x) + h(x) \right) dx$$

$$= \int_a^b \left(w(x) f(x) g(x) + w(x) f(x) h(x) \right) dx$$

$$= \int_a^b w(x) f(x) g(x)\, dx + \int_a^b w(x) f(x) h(x)\, dx$$

$$= \langle f, g \rangle + \langle f, h \rangle.$$

P3 $r\langle f,g\rangle = r\int_a^b w(x)f(x)g(x)\,dx$

$= \int_a^b w(x)(rf(x))g(x)\,dx = \langle rf, g\rangle$

$= \int_a^b w(x)f(x)(rg(x))\,dx = \langle f, rg\rangle.$

P4 $\langle f,f\rangle = \int_a^b w(x)f(x)f(x)\,dx = \int_a^b w(x)(f(x))^2\,dx$

≥ 0 since $w(x)(f(x))^2 \geq 0$ for $x \in [a,b]$.

Finally, $\langle f,f\rangle = \int_a^b w(x)(f(x))^2\,dx = 0$

if and only if $w(x)(f(x))^2 = 0$ for all $x \in [a,b]$.
Notice that if $w(x)(f(x))^2 > 0$ for some $x \in [a,b]$ then, since $w(x)(f(x))^2$ is continuous, there is some interval $[c,d] \subset [a,b]$ such that $w(x)(f(x))^2 > 0$ for $x \in [c,d]$ and then

$$\int_a^b w(x)(f(x))^2\,dx \geq \int_c^d w(x)(f(x))^2\,dx > 0$$

(and conversely). ∎