SECTION 4.1
EXERCISE #27

4.1.27 Find the area of the triangle in $\mathbb{R}^3$ with vertices $(2, 1, -3)$, $(3, 0, 4)$, and $(1, 0, 5)$.

Solution

Let's introduce vectors $\vec{b}$ and $\vec{c}$ where $\vec{b}$ is from $(2, 1, -3)$ to $(3, 0, 4)$ and $\vec{c}$ is from $(2, 1, -3)$ to $(2, 0, 5)$. So

$\vec{b} = [3-2,\ 0-1,\ 4-(-3)] = [1, -1, 7]$, and

$\vec{c} = [1-2,\ 0-1,\ 5-(-3)] = [-1, -1, 8]$.

By Theorem 4.1.A, the area of the parallelogram determined by $\vec{b}$ and $\vec{c}$ is $\|\vec{b} \times \vec{c}\|$, so the area of the desired triangle is

$$\frac{1}{2}\|\vec{b} \times \vec{c}\| = \frac{1}{2}\left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 7 \\ -1 & -1 & 8 \end{vmatrix} \right\|$$

$$= \frac{1}{2}\Big\| \big((-1)(8) - (7)(-1)\big)\hat{i} - \big((1)(8) - (7)(-1)\big)\hat{j} + \big((1)(-1) - (-1)(-1)\big)\hat{k} \Big\|$$

$$= \frac{1}{2}\| \hat{i} - 15\hat{j} - 2\hat{k} \|$$

$$= \frac{1}{2}\sqrt{(1)^2 + (-15)^2 + (-2)^2} = \frac{1}{2}\sqrt{1 + 225 + 4}$$

$$= \boxed{\frac{\sqrt{230}}{2}}. \quad \square$$