SECTION 4.1
EXERCISE #57

4.1.57 Prove Theorem 4.1(2): For $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, $\vec{a} \times (\vec{b} \times \vec{c})$ is not, in general, the same as $(\vec{a} \times \vec{b}) \times \vec{c}$.

Proof

Notice that

$$\hat{i} \times \hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0\hat{i} - 0\hat{j} + 0\hat{k} = \vec{0}.$$

Similarly, $\hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$. We take advantage of this to show the nonassociativity of $\times$. Consider $\vec{a} = \vec{b} = \hat{i}$ and $\vec{c} = \hat{j}$. Then

$$\vec{a} \times (\vec{b} \times \vec{c}) = \hat{i} \times (\hat{i} \times \hat{j}) = \hat{i} \times \hat{k} = -\hat{j}$$

since

$$\hat{i} \times \hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0\hat{i} - 0\hat{j} + 1\hat{k} = \hat{k} \text{ and}$$

$$\hat{i} \times \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0\hat{i} - (1)\hat{j} + 0\hat{k} = -\hat{j}.$$

But $(\vec{a} \times \vec{b}) \times \vec{c} = (\hat{i} \times \hat{i}) \times \hat{j} = \vec{0} \times \hat{j} = \vec{0}$. So $\hat{i} \times (\hat{i} \times \hat{j}) \neq (\hat{i} \times \hat{i}) \times \hat{j}$ and $\vec{a} \times (\vec{b} \times \vec{c})$ is not, in general, the same as $(\vec{a} \times \vec{b}) \times \vec{c}$. ∎