

SECTION 4.1
EXERCISE #59

4.1, 59 Prove Theorem 4.1(b): For $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$,
 $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} = \pm$ volume of the box
determined by \vec{a}, \vec{b} , and \vec{c} .

Proof

By Theorem 4.1.B, the volume of the box is
 $|\vec{a} \cdot (\vec{b} \times \vec{c})|$. So the result will follow if

$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$. We have

$$\begin{aligned} (\vec{a} \times \vec{b}) \cdot \vec{c} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot (c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k}) \\ &= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k} \right) \cdot (c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k}) \\ &= ((a_2 b_3 - a_3 b_2) \hat{i} - (a_1 b_3 - a_3 b_1) \hat{j} + (a_1 b_2 - a_2 b_1) \hat{k}) \cdot (c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k}) \\ &= (a_2 b_3 - a_3 b_2) c_1 - (a_1 b_3 - a_3 b_1) c_2 + (a_1 b_2 - a_2 b_1) c_3 \\ &= a_1 (b_2 c_3 - b_3 c_2) + a_2 (b_3 c_1 - b_1 c_3) + a_3 (b_1 c_2 - b_2 c_1) \\ &= (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \cdot ((b_2 c_3 - b_3 c_2) \hat{i} - (b_1 c_3 - b_3 c_1) \hat{j} + (b_1 c_2 - b_2 c_1) \hat{k}) \\ &= (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \cdot \left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \hat{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \hat{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \hat{k} \right) \\ &= (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \vec{a} \cdot (\vec{b} \times \vec{c}). \end{aligned}$$

So the necessary equality holds. ■