SECTION 4.2
EXERCISE #33

4.2.33 Without using the multiplicative property of determinants (Theorem 4.4), prove that $\det(AB) = \det(A)\det(B)$ for the case where B is a diagonal matrix.

<u>Proof</u>

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{bmatrix}$.

Then

$(AB)^T = B^T A^T$ by Note 1.3.B

$$= \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} b_1 a_{11} & b_1 a_{21} & \cdots & b_1 a_{n1} \\ b_2 a_{12} & b_2 a_{22} & \cdots & b_2 a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ b_n a_{1n} & b_n a_{2n} & \cdots & b_n a_{nn} \end{bmatrix}.$$

So by Theorem 4.2A(4), "The Scalar Multiplication Property,"

$\det((AB)^T) = b_1 b_2 \cdots b_n \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix}$. Since $B = B^T$

is upper triangular then $\det(B) = \det(B^T) = b_1 b_2 \cdots b_n$ by Page 255 Example 4, so $\det((AB)^T) = \det(B)\det(A^T)$.
By Theorem 4.2.A(2), "The Transpose Property," $\det(A^T) = \det(A)$ and $\det((AB)^T) = \det(AB)$, hence

$$\det((AB)^T) = \det(B)\det(A^T) \text{ implies}$$

$$\det(AB) = \det(A)\det(B), \text{ as claimed. } \blacksquare$$