SECTION 4.2
EXERCISE # 35

4.2.35 Prove that, if three $n \times n$ matrices A, B, and C are identical except for the kth rows $\vec{a}_k$, $\vec{b}_k$, and $\vec{c}_k$, respectively, which are related by $\vec{a}_k = \vec{b}_k + \vec{c}_k$, then $\det(A) = \det(B) + \det(C)$.

Proof

Let $\vec{a}_k = [a_{k1}, a_{k2}, \dots, a_{kn}]$, $\vec{b}_k = [b_{k1}, b_{k2}, \dots, b_{kn}]$, and $\vec{c}_k = [c_{k1}, c_{k2}, \dots, c_{kn}]$. Since A, B, and C have the same entries except along row k then the following minor matrices are the same:

$A_{k1} = B_{k1} = C_{k1}$; $A_{k2} = B_{k2} = C_{k2}, \dots, A_{kn} = B_{kn} = C_{kn}$.

So the following cofactors are the same:

$a'_{k1} = b'_{k1} = c'_{k1}$, $a'_{k2} = b'_{k2} = c'_{k2}, \dots, a'_{kn} = b'_{kn} = c'_{kn}$.

By Theorem 4.2, "General Expansion by Minors", we can find determinants by expanding along the kth rows to get

$$\det(A) = a_{k1}a'_{k1} + a_{k2}a'_{k2} + \dots + a_{kn}a'_{kn},$$

$$\det(B) = b_{k1}b'_{k1} + b_{k2}b'_{k2} + \dots + b_{kn}b'_{kn}$$
$$= b_{k1}a'_{k1} + b_{k2}a'_{k2} + \dots + b_{kn}a'_{kn}, \text{ and}$$

$$\det(C) = c_{k1}c'_{k1} + c_{k2}c'_{k2} + \dots + c_{kn}c'_{kn}$$
$$= c_{k1}a'_{k1} + c_{k2}a'_{k2} + \dots + c_{kn}a'_{kn}.$$

Since $\vec{a}_k = \vec{b}_k + c_k$ then $a_{kj} = b_{kj} + c_{kj}$ for $j = 1, 2, \dots, n$ and so

$$\det(A) = a_{k1}a'_{k1} + a_{k2}a'_{k2} + \dots + a_{kn}a'_{kn}$$
$$= (b_{k1} + c_{k1})a'_{k1} + (b_{k2} + c_{k2})a'_{k2} + \dots + (b_{kn} + c_{kn})a'_{kn}$$
$$= (b_{k1}a'_{k1} + b_{k2}a'_{k2} + \dots + b_{kn}a'_{kn})$$
$$+ (c_{k1}a'_{k1} + c_{k2}a'_{k2} + \dots + c_{kn}a'_{kn}) = \det(B) + \det(C). \blacksquare$$