SECTION 4.3

EXERCISE 31

4.3.31 Solve the system of linear equations by Cramer's Rule:

$$3x_1 + 2x_2 - x_3 = 1$$
$$x_1 - 4x_2 + x_3 = -2$$
$$5x_1 + 2x_2 \quad = 1.$$

Solution

By Cramer's Rule (Theorem 4.5), we have $x_k = \dfrac{\det(B_k)}{\det(A)}$ for $k = 1, 2, 3$ where B_k is the matrix obtained from matrix A by replacing the kth column of A by the column vector $\vec{b}$ in $A\vec{x} = \vec{b}$. Here, we have

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -4 & 1 \\ 5 & 2 & 0 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

So, using row operations and Theorem 4.2.A, "Properties of Determinants," we have

$$\det(A) = \begin{vmatrix} 3 & 2 & -1 \\ 1 & -4 & 1 \\ 5 & 2 & 0 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_2} -\begin{vmatrix} 1 & -4 & 1 \\ 3 & 2 & -1 \\ 5 & 2 & 0 \end{vmatrix} \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 3R_1}$$

$$-\begin{vmatrix} 1 & -4 & 1 \\ 0 & 14 & -4 \\ 0 & 22 & -5 \end{vmatrix} = -\left((1)\begin{vmatrix} 14 & -4 \\ 22 & -5 \end{vmatrix} - 0 + 0\right) = -\left((14)(-5) - (-4)(22)\right)$$
$$= -70 + 88 = -18.$$

Next, $B_1 = \begin{vmatrix} 1 & 2 & -1 \\ -2 & -4 & 1 \\ 1 & 2 & 0 \end{vmatrix}$ and $\det(B_1) = \begin{vmatrix} 1 & 2 & -1 \\ -2 & -4 & 1 \\ 1 & 2 & 0 \end{vmatrix}$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + 2R_1} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{vmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{vmatrix} = 0,$$

$$B_2 = \begin{bmatrix} 3 & 1 & -1 \\ 1 & -2 & 1 \\ 5 & 1 & 0 \end{bmatrix} \text{ and } \det(B) = \begin{vmatrix} 3 & 1 & -1 \\ 1 & -2 & 1 \\ 5 & 1 & 0 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_2}$$

$$= -\begin{vmatrix} 1 & -2 & 1 \\ 3 & 1 & -1 \\ 5 & 1 & 0 \end{vmatrix} \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 3R_1} -\begin{vmatrix} 1 & -2 & 1 \\ 0 & 7 & -4 \\ 0 & 11 & -5 \end{vmatrix}$$

SECTION 4.3
EXERCISE 31 (continued)

$$= -\left((2)\begin{vmatrix} 7 & -4 \\ 11 & -5 \end{vmatrix} - 0 + 0\right) = -\left((7)(-5) - (-4)(11)\right)$$

$$= -(-35+44) = -9, \text{ and}$$

$B_3 = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -4 & -2 \\ 5 & 2 & 1 \end{bmatrix}$ so that $\det(B_3) = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -4 & -2 \\ 5 & 2 & 1 \end{vmatrix}$

$$\xrightarrow{R_1 \leftrightarrow R_2} -\begin{vmatrix} 1 & -4 & -2 \\ 3 & 2 & 1 \\ 5 & 2 & 1 \end{vmatrix} \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 3R_1} -\begin{vmatrix} 1 & -4 & -2 \\ 0 & 14 & 7 \\ 0 & 22 & 11 \end{vmatrix}$$

$$= -\left(\left((14)(11) - (7)(22)\right) - 0 + 0\right) = -(154 - 154) = 0.$$

We then have by Cramer's Rule

$$x_1 = \frac{\det(B_1)}{\det(A)} = \frac{0}{-18} = 0$$

$$x_2 = \frac{\det(B_2)}{\det(A)} = \frac{-9}{-18} = \frac{1}{2}$$

$$x_3 = \frac{\det(B_3)}{\det(A)} = \frac{0}{-18} = 0.$$

So, $\boxed{x_1 = 0,\ x_2 = \frac{1}{2},\ x_3 = 0.}$ $\square$