SECTION 4.3

NUMBER 37

4.3.37 Prove that a square matrix is invertible if and only if its adjoint is an invertible matrix.

<u>Proof</u>

Suppose A is invertible. Then by Theorem 4.3, "Determinant Criterion for Invertibility," $\det(A) \neq 0$. By Theorem 4.6, "Property of the Adjoint,"
$\operatorname{adj}(A)A = \det(A)I$. So

$$\operatorname{adj}(A)\left(\frac{1}{\det(A)}A\right) = I$$

and $(\operatorname{adj}(A))^{-1} = \frac{1}{\det(A)}A$ (we may want to reference Theorem 1.21, "A Commutative Property," here), so that $\operatorname{adj}(A)$ is invertible.

Now, suppose $\operatorname{adj}(A)$ is invertible. Again by Theorem 4.6, $\operatorname{adj}(A)A = \det A\, I$ and so $A = \det(A)\operatorname{adj}(A)^{-1}$. This equation shows that if $\det(A) = 0$ then $A = 0$; but then $\operatorname{adj}(A) = 0$ and $\operatorname{adj}(A)$ is singular, contradicting the hypothesis that $\operatorname{adj}(A)$ is invertible. So it must be that $\det(A) \neq 0$ and hence by Theorem 4.3, A is invertible. ∎