SECTION 4.3
NUMBER 39

4.3.39 Let A be an invertible $n \times n$ matrix with $n > 1$. Prove that $\mathrm{adj}(\mathrm{adj}(A)) = (\det(A))^{n-2} A$.
HINT: Use Exercises 37 and 38 which show that A is invertible if and only if $\mathrm{adj}(A)$ is invertible, and $\det(\mathrm{adj}(A)) = (\det(A))^{n-1}$.

Proof

By Theorem 4.6, "Property of the Adjoint," $B\,\mathrm{adj}(B) = \det(B)\,I$ for any square matrix B. With $B = \mathrm{adj}(A)$ we have

$$\mathrm{adj}(A)\ \mathrm{adj}(\mathrm{adj}(A)) = \det(\mathrm{adj}(A))\,I. \quad (*)$$

Since A is invertible by hypothesis then by Exercise 37 $\mathrm{adj}(A)$ is invertible. So we have from $(*)$ that

$$\begin{aligned}\mathrm{adj}(\mathrm{adj}(A)) &= \mathrm{adj}(A)^{-1} \det(\mathrm{adj}(A))\,I \\ &= \det(\mathrm{adj}(A))\ \mathrm{adj}(A)^{-1}. \quad (**)\end{aligned}$$

Again by Theorem 4.6 (since $\det(A) \neq 0$ by Theorem 4.3, "Determinant Criterion for Invertibility") $\mathrm{adj}(A)^{-1} = \frac{1}{\det(A)} A$, so from $(**)$,

$$\begin{aligned}\mathrm{adj}(\mathrm{adj}(A)) &= \det(\mathrm{adj}(A)) \frac{1}{\det(A)} A \\ &= \det(A)^{n-1} \frac{1}{\det(A)} A \quad \text{by Exercise 38} \\ &= \det(A)^{n-2} A,\end{aligned}$$

as claimed. ∎