SECTION 5.1
EXERCISE 11

5.1.11 Find the characteristic polynomial, the real eigenvalues, and the corresponding eigenvectors of

$$A = \begin{bmatrix} -2 & 0 & 0 \\ -5 & -2 & -5 \\ 5 & 0 & 3 \end{bmatrix}.$$

Solution

First, the characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 0 & 0 \\ -5 & -2-\lambda & -5 \\ 5 & 0 & 3-\lambda \end{vmatrix}$$

$$= (-2-\lambda)\begin{vmatrix} -2-\lambda & -5 \\ 0 & 3-\lambda \end{vmatrix} - 0 + 0$$

$$= (-2-\lambda)\big((-2-\lambda)(3-\lambda) - (-5)(0)\big) = \boxed{(-2-\lambda)^2(3-\lambda).}$$

For the eigenvalues, we set $p(\lambda) = 0$ to get $(-2-\lambda)^3(3-\lambda) = 0$ and so the eigenvalues are $\boxed{\lambda_1 = -2 \text{ and } \lambda_2 = 3.}$ For the eigenvectors, we consider the system of equations $(A - \lambda I)\vec{v} = \vec{0}$:

$\underline{\lambda_1 = -2}$ $(A - (-2)I)\vec{v} = \vec{0}$ has associated augmented matrix

$$\left[\begin{array}{ccc|c} -2-(-2) & 0 & 0 & 0 \\ -5 & -2-(-2) & -5 & 0 \\ 5 & 0 & 3-(-2) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -5 & 0 & -5 & 0 \\ 5 & 0 & 5 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 5 & 0 & 5 & 0 \\ -5 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 5 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1/5} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$ and the associated system of equations is

$$\begin{aligned} v_1 + v_3 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} v_1 &= -v_3 \\ v_2 &= v_2 \\ v_3 &= v_3. \end{aligned}$$

Let $r = v_2$ and $s = v_3$ be free

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EXERCISE 11 (continued)

variables so that $v_1 = -r$ where $r, s \in \mathbb{R}$.

$v_2 = s$

$v_3 = r$

So the eigenvectors of A associated with $\lambda_1 = -2$ are

$$\boxed{\vec{v}_1 = r\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ where } r, s \in \mathbb{R}, \text{ and not both } r \text{ and } s \text{ are } 0.}$$

$\lambda_2 = 3$ $(A - 3I)\vec{v} = \vec{0}$ has associated augmented matrix

$$\left[\begin{array}{ccc|c} -2-(3) & 0 & 0 & 0 \\ -5 & -2-(3) & -5 & 0 \\ 5 & 0 & 3-(3) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -5 & 0 & 0 & 0 \\ -5 & -5 & -5 & 0 \\ 5 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{R_3 \to R_3 + R_1}{\xrightarrow{R_2 \to R_2 - R_1}} \left[\begin{array}{ccc|c} -5 & 0 & 0 & 0 \\ 0 & -5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{R_2 \to R_2/(-5)}{\xrightarrow{R_1 \to R_1/(-5)}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

and the associated system of equations is

$$\begin{aligned} v_1 &= 0 \\ v_2 + v_3 &= 0 \\ 0 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} v_1 &= 0 \\ v_2 &= -v_3 \\ v_3 &= v_3. \end{aligned}$$

Let $t = v_3$ be a free variable so that

$v_1 = 0$ where $t \in \mathbb{R}$. So the eigenvalues of A associated with $\lambda_2 = 3$ are

$v_2 = -t$

$v_3 = t$

$$\boxed{\vec{v}_2 = t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \text{ where } t \in \mathbb{R}, t \neq 0.}$$

□