SECTION 5.1
EXERCISE 15

5.1.15 Find characteristic polynomial, the real eigenvalues, and the corresponding eigenvectors of

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -2 & 1 \\ 2 & 0 & -1 \end{bmatrix}.$$

Solution

First, the characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 0-\lambda & 0 & 1 \\ -2 & -2-\lambda & 1 \\ 2 & 0 & -1-\lambda \end{vmatrix}$$

$$= (-\lambda)\begin{vmatrix} -2-\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} - 0 + (1)\begin{vmatrix} -2 & -2-\lambda \\ 2 & 0 \end{vmatrix}$$

$$= -\lambda\big((-2-\lambda)(-1-\lambda) - (1)(0)\big) + \big((-2)(0) - (-2-\lambda)(2)\big)$$

$$= -\lambda(-2-\lambda)(-1-\lambda) + 2(2+\lambda)$$

$$= \lambda(2+\lambda)(-1-\lambda) + 2(2+\lambda) = (2+\lambda)(\lambda(-1-\lambda)+2)$$

$$= (2+\lambda)(-\lambda-\lambda^2+2) = (2+\lambda)(-\lambda+1)(\lambda+2)$$

$$= \boxed{(\lambda+2)^2(-\lambda+1)}.$$

For the eigenvalues, we set $p(\lambda) = 0$ to get $(\lambda+2)^2(-\lambda+1) = 0$ and so the eigenvalues are $\boxed{\lambda_1 = -2 \text{ and } \lambda_1 = 1.}$ For the eigenvectors, we

consider the system of equations $(A - \lambda I)\vec{v} = \vec{0}$:

$\lambda_1 = -2$ $\quad (A-(-2)I)\vec{v} = \vec{0}$ has associated augmented matrix

$$\left[\begin{array}{ccc|c} 0-(-2) & 0 & 1 & 0 \\ -2 & -2-(-2) & 1 & 0 \\ 2 & 0 & -1-(-2) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2 \to R_2/2]{R_1 \to R_1/2} \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - \frac{1}{2}R_2}$$

SECTION 5.1

EXERCISE 15 (continued)

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, and the associated system of equations is

$$\begin{aligned} v_1 &= 0 \\ v_3 &= 0 \\ 0 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} v_1 &= 0 \\ v_2 &= v_2 \\ v_3 &= 0. \end{aligned}$$

Let $r = v_2$ be a free variable so that

$$\begin{aligned} v_1 &= 0 \\ v_2 &= r \\ v_3 &= 0 \end{aligned}$$

where $r \in \mathbb{R}$. So the eigenvectors of A associated with $\lambda_1 = -2$ are

$$\boxed{\vec{v}_1 = r \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ where } r \in \mathbb{R},\ r \neq 0.}$$

$\lambda_2 = 1$ $(A - (1)I)\vec{v} = \vec{0}$ has associated augmented matrix

$$\left[\begin{array}{ccc|c} 0-(1) & 0 & 1 & 0 \\ -2 & -2-(1) & 1 & 0 \\ 2 & 0 & -1-(1) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ -2 & -3 & 1 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2 \to R_2/(-3)]{R_1 \to -R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

and the augmented system of equations is

$$\begin{aligned} v_1 \quad - v_3 &= 0 \\ v_2 + \tfrac{1}{3} v_3 &= 0 \\ 0 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} v_1 &= v_3 \\ v_2 &= \tfrac{1}{3} v_3 \\ v_3 &= v_3. \end{aligned}$$

Let $s = v_3/3$ be a free variable so that

$$\begin{aligned} v_1 &= 3s \\ v_2 &= s \\ v_3 &= 3s \end{aligned}$$

where $s \in \mathbb{R}$. So the eigenvectors of A associated with $\lambda_2 = -1$ are

$$\boxed{\vec{v}_2 = s \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \text{ where } s \in \mathbb{R},\ s \neq 0.}$$

□