SECTION 5.2
NUMBER 17

5.2.17 Find the eigenvalues and eigenvectors for $T([x, y]) = [2x-3y, -3x+2y]$.

<u>Solution</u>

Well, lets find the standard matrix representation for T:

$$T(\hat{i}) = T([1,0]) = [2,-3]$$

$$T(\hat{j}) = T([0,1]) = [-3, 2],$$

and

$$A_T = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}.$$

Now the eigenvalues and eigenvectors of T are the same (by definition) as those of A_T.

For eigenvalues, consider

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -3 \\ -3 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - (-3)(-3)$$

$$= 4 - 4\lambda + \lambda^2 - 9 = \lambda^2 - 4\lambda - 5 = (\lambda+1)(\lambda-5)$$

(this is characteristic polynomial).

For eigenvalues, set $(\lambda+1)(\lambda-5) \equiv 0$ and then $\boxed{\lambda_1 = -1 \text{ and } \lambda_2 = 5 \text{ are the eigenvalues.}}$

For eigenvectors:

<u>$\lambda_1 = -1$</u> Consider the system of equations $(A - \lambda I)\vec{v} = \vec{0}$. This gives

$$[A - \lambda I \mid \vec{0}] = \left[\begin{array}{cc|c} 2-(-1) & -3 & 0 \\ -3 & 2-(-1) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 3 & -3 & 0 \\ -3 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{cc|c} 3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1/(3)} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

This gives $v_1 - v_2 = 0$ or $v_1 = v_2$ or
$0 = 0$ $\quad v_2 = v_2$
with $r = v_2$ as a free variable and then
$v_1 = r$ where $r \in \mathbb{R}$. So the eigenvectors
$v_2 = r$
for $\lambda_1 = -1$ are of the form $\vec{v}_1 = \begin{bmatrix} r \\ r \end{bmatrix} = r\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $r \in \mathbb{R}$, $r \neq 0$.

$\underline{\lambda_2 = 5}$ Consider $\left[\begin{array}{cc|c} 2-(5) & -3 & 0 \\ -3 & 2-(5) & 0 \end{array}\right] = \left[\begin{array}{cc|c} -3 & -3 & 0 \\ -3 & -3 & 0 \end{array}\right]$

$\xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} -3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1/(-3)} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$.

This gives $v_1 + v_2 = 0$ or $v_1 = -v_2$ or
$0 = 0$ $\quad v_2 = v_2$
with $s = v_2$ as a free variable,
$v_1 = -s$ where $s \in \mathbb{R}$. So the eigenvectors
$v_2 = s$
for $\lambda_2 = 5$ are of the form $\vec{v}_2 = \begin{bmatrix} -s \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ where $s \in \mathbb{R}$, $s \neq 0$.

$\square$

$\underline{\text{Note}}$ We can diagonalize A (see § 5.2) as $AC = CD$ where

$$D = \begin{bmatrix} \lambda_1 & \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \text{ and } C = [\vec{v}_1 \; \vec{v}_2]$$
$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$