SECTION 5.1
EXERCISE #25

5.1.25 Prove that if A is a square matrix, then AA^T and A^TA have the same eigenvalues.

<u>Proof</u>

Let λ be an eigenvalue for AA^T. Then $AA^T\vec{v} = \lambda\vec{v}$ for some nonzero vector $\vec{v}$. Let $\vec{w} = A^T\vec{v}$. Then
$(A^TA)\vec{w} = (A^TA)(A^T\vec{v}) = A^T(AA^T\vec{v}) = A^T(\lambda\vec{v}) = \lambda A^T\vec{v} = \lambda\vec{w}$

and so λ is an eigenvalue of A^TA (with corresponding eigenvector $A^T\vec{v}$ where $\vec{v}$ is an eigenvector of AA^T corresponding to eigenvalue λ of AA^T).
So every eigenvalue of AA^T is an eigenvalue of A^TA.

Next, let λ be an eigenvalue for A^TA.
Then $A^TA\vec{v} = \lambda\vec{v}$ for some nonzero vector $\vec{v}$.
Let $\vec{w} = A\vec{v}$. Then
$(AA^T)\vec{w} = (AA^T)(A\vec{v}) = A(A^TA\vec{v}) = A(\lambda\vec{v}) = \lambda A\vec{v} = \lambda\vec{w}$
and so λ is an eigenvalue of AA^T (with corresponding eigenvector $A\vec{v}$ where $\vec{v}$ is an eigenvector of A^TA corresponding to eigenvalue λ of A^TA). So every eigenvalue of A^TA is an eigenvalue of AA^T.

Therefore the eigenvalues of AA^T are the same as the eigenvalues of A^TA. ∎