SECTION 5.1
EXERCISE # 27

5.1, 27 Prove Theorem 5.1(1): Let A be an $n \times n$ matrix. If λ is an eigenvalue of A with $\vec{v}$ as a corresponding eigenvector then λ^k is an eigenvalue of A^k again with $\vec{v}$ as a corresponding eigenvector, for any positive integer k.

<u>Proof 1</u>

We have

$$A^k \vec{v} = \underbrace{A(A(\cdots A}_{k \text{ times}}\vec{v})\cdots))$$

$$= \underbrace{A A \cdots A}_{k-1 \text{ times}} \lambda \vec{v} = \lambda A^{k-1} \vec{v}$$

$$= \lambda \underbrace{A(A(\cdots A}_{k-1 \text{ times}}\vec{v})\cdots)) = \lambda \underbrace{AA\cdots A}_{k-2 \text{ times}} \lambda \vec{v} = \lambda^2 A^{k-2} \vec{v}$$

$$= \cdots = \lambda^{k-1} A\vec{v} = \lambda^{k-1} \lambda \vec{v} = \lambda^k \vec{v}.$$

So λ^k is an eigenvalue of A^k with corresponding eigenvector $\vec{v}$. ∎

<u>Proof 2</u>

A cleaner proof can be given using mathematical induction (see Appendix A). With $k=1$, we have $\lambda^1 = \lambda$ is an eigenvalue of A with corresponding eigenvector $\vec{v}$ by hypothesis. Suppose the result holds for $k=i$. That is, λ^i is an eigenvalue of A^i with corresponding eigenvector $\vec{v}$; that is, $A^i \vec{v} = \lambda^i \vec{v}$. Consider $k = i+1$. We have

$$A^k \vec{v} = A^{i+1} \vec{v} = A(A^i \vec{v}) = A(\lambda^i \vec{v}) \text{ by the induction hypothesis}$$
$$= \lambda^i (A\vec{v}) = \lambda^i (\lambda \vec{v}) = \lambda^{i+1} \vec{v}.$$

So if the result holds for $k = i$ then it holds for $k = i+1$. Then, by mathematical induction, the result holds for all k a positive integer. ∎