SECTION 5.1
EXERCISE #29

5.1.29 Prove Theorem 5.1(3): Let A be an $n \times n$ matrix. If λ is an eigenvalue of A, then the set E_λ consisting of the zero vector together with all eigenvectors of A for this eigenvalue λ is a subspace of n-space, the eigenspace of λ.

<u>Proof</u>

We apply Theorem 3.2, "Test for a Subspace." First, the zero vector is in E_λ so E_λ is nonempty.

Let $\vec{v}$ and $\vec{w}$ be in E_λ. That is, $A\vec{v} = \lambda\vec{v}$ and $A\vec{w} = \lambda\vec{w}$. Then

$$
\begin{aligned}
A(\vec{v}+\vec{w}) &= A\vec{v} + A\vec{w} && \text{by Theorem 1.3.A(10), "Distributive Laws of Matrix Multiplication"} \\
&= \lambda\vec{v} + \lambda\vec{w} \\
&= \lambda(\vec{v}+\vec{w}) && \text{by Theorem 1.3.A(4), "Left Distribution Law".}
\end{aligned}
$$

So $\vec{v}+\vec{w}$ is an eigenvector of A (if it is nonzero; if it is the zero vector then it is in E_λ by hypothesis) with corresponding eigenvalue λ. That is, $\vec{v}+\vec{w} \in E_\lambda$.

Let r be a scalar and $\vec{v} \in E_\lambda$. If $r = 0$ then $r\vec{v} = \vec{0} \in E_\lambda$ by hypothesis. If $r \neq 0$ then

$$
\begin{aligned}
A(r\vec{v}) &= r(A\vec{v}) && \text{by Theorem 1.3.A(7), "Scalars Pull Through"} \\
&= r\lambda\vec{v} && \text{since } \vec{v} \in E_\lambda \text{ and so } A\vec{v} = \lambda\vec{v} \\
&= \lambda(r\vec{v}) && \text{by Theorem 1.3.A(6), "Associative Law of Scalar Multiplication".}
\end{aligned}
$$

So $r\vec{v}$ is an eigenvector of A with corresponding eigenvalue λ. That is, $r\vec{v} \in E_\lambda$.

So by Theorem 3.2, E_λ is a subspace of $\mathbb{R}^n$. ■