SECTION 5.1
EXERCISE #35

5.1.35 Principle of Biorthogonality Let A be an $n \times n$ real matrix. Let $\vec{v}$ in $\mathbb{R}^n$ be an eigenvector of A with corresponding eigenvalue λ, and let $\vec{w}$ in $\mathbb{R}^n$ be an eigenvector of A^T with corresponding eigenvalue α. Prove that if $\lambda \neq \alpha$, then $\vec{v}$ and $\vec{w}$ are perpendicular vectors.

Proof

We are given $A\vec{v} = \lambda\vec{v}$, $A^T\vec{w} = \alpha\vec{w}$, and $\lambda \neq \alpha$. Notice that $\vec{v}$ and $\vec{w}$ are column vectors so $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v} = \vec{w}^T\vec{v}$. Now

$(\vec{w}^T A)\vec{v} = (A^T\vec{w})^T\vec{v} = (\alpha\vec{w})^T\vec{v} = \alpha\vec{w}^T\vec{v} = \alpha(\vec{v} \cdot \vec{w})$ and

$(\vec{w}^T A)\vec{v} = \vec{w}^T(A\vec{v}) = \vec{w}^T(\lambda v) = \lambda\vec{w}^T\vec{v} = \lambda(\vec{v} \cdot \vec{w})$.

So $\alpha(\vec{v} \cdot \vec{w}) = \lambda(\vec{v} \cdot \vec{w})$ or $\alpha(\vec{v} \cdot \vec{w}) - \lambda(\vec{v} \cdot \vec{w}) = 0$ or $(\alpha - \lambda)(\vec{v} \cdot \vec{w}) = 0$. Since $\lambda \neq \alpha$ then it must be that $\vec{v} \cdot \vec{w} = 0$; that is, $\vec{v}$ and $\vec{w}$ are perpendicular. ∎