SECTION 5.1

EXERCISE # 41

5.1.41 Let $\vec{v}_1$ and $\vec{v}_2$ be eigenvectors of a linear transformation $T: V \to V$ with corresponding eigenvalues λ_1 and λ_2, respectively. Prove that, if $\lambda_1 \neq \lambda_2$, then $\vec{v}_1$ and $\vec{v}_2$ are independent vectors.

Proof

Consider $r_1\vec{v}_1 + r_2\vec{v}_2 = \vec{0}$. Since T is linear then $T(\vec{0}) = \vec{0}$ (see the Note after the example Page 153 Number 32 in the class notes for Section 2.3, "Transformations of Euclidean Space").

So

$\vec{0} = T(\vec{0}) = T(r_1\vec{v}_1 + r_2\vec{v})$

$= r_1 T(\vec{v}_1) + r_2 T(\vec{v}_2)$ since T is linear

$= r_1(\lambda_1\vec{v}_1) + r_2(\lambda_2\vec{v}_2)$ since $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors corresponding to eigenvalues λ_1 and λ_2.

Since $r_1\vec{v}_1 + r_2\vec{v}_2 = \vec{0}$ then $\lambda_1(r_1\vec{v}_1 + r_2\vec{v}_2) = \lambda_1\vec{0} = \vec{0}$.

So we have $r_1\lambda_1\vec{v}_1 + r_2\lambda_2\vec{v}_2 = \vec{0}$

and $r_1\lambda_1\vec{v}_1 + r_2\lambda_1\vec{v}_2 = \vec{0}$.

Subtracting we have $r_2(\lambda_1 - \lambda_2)\vec{v}_2 = \vec{0}$.

Since $\lambda_1 \neq \lambda_2$ by hypothesis and $\vec{v}_2 \neq \vec{0}$ since it is an eigenvector, then it must be that $r_2 = 0$. With $r_2 = 0$ in the original equation $r_1\vec{v}_1 + r_2\vec{v}_2 = \vec{0}$ we have that $r_1\vec{v}_1 = \vec{0}$; since $\vec{v}_1 \neq \vec{0}$ because it is an eigenvector then it must be that $r_1 = 0$.

Since $r_1 = r_2 = 0$ is necessary then $\vec{v}_1$ and $\vec{v}_2$ are independent vectors. ∎