SECTION 5.2
EXERCISE #19

5.2.19 Let A be an $n \times n$ matrix.

(a) Prove that if A is similar to rA where r is a real scalar other than 1 or -1, then all eigenvalues of A are zero.

Proof

If $r=0$ then $A = C^{-1}OC = O$ (for some invertible matrix C) and so $\det(A - \lambda cI) = \lambda^n$. So all eigenvalues of A are zero and the result holds for $r=0$. So we now consider the case $r \neq 0$.

If λ is an eigenvalue of A with corresponding eigenvector $\vec{v}$. Then $A\vec{v} = \lambda\vec{v}$ and so $(rA)\vec{v} = r(A\vec{v}) = r(\lambda\vec{v}) = (r\lambda)\vec{v}$. Hence $r\lambda$ is an eigenvalue of rA. If λ is an eigenvalue of rA with corresponding eigenvector $\vec{v}$ then
$A\vec{v} = (r/r)A\vec{v} = (1/r)(rA)\vec{v} = (1/r)\lambda\vec{v} = (\lambda/r)\vec{v}$.
Hence λ/r is an eigenvalue of A. That is, λ is an eigenvalue of A if and only if $r\lambda$ is an eigenvalue of rA.

Let λ_M be an eigenvalue of maximum magnitude (λ_M may be complex and "magnitude" is then the modulus of λ_M, $|\lambda_M|$; see Section 9.1, "Algebra of Complex Numbers"). Then $r\lambda_M$ is an eigenvalue of rA of maximum magnitude. By Exercise #18, since A is similar to rA, A and rA have the same eigenvalues. So the eigenvalues of largest magnitude must be of the same size and so $|\lambda_M| = |r\lambda_M| = |r||\lambda_M|$. So either $|r|=1$ or $|\lambda_M|=0$. Since r is real and $r \neq \pm 1$ then $|r| \neq 1$ and so $|\lambda_M| = 0$; that is, $\lambda_M = 0$. Since the largest magnitude eigenvalue of A is 0 then all eigenvalues of A are 0, as claimed. ■

$\longrightarrow$

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(b) What can you say about A if it is diagonalizable and similar to rA for some real r where $|r| \neq 1$?

Solution

Since A is similar to rA where $|r| \neq 1$ then by Part (a), all eigenvalues of A are 0. Since A is diagonalizable then for some invertible C, $C^{-1}AC = D$ where D is a diagonal matrix (by Definition 5.3, "Diagonalizable Matrix"). Then $AC = CD$ and so by Theorem 5.2 "Matrix Summary of Eigenvalues of A,"
$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = 0$$
(where 0 represents the $n \times n$ zero matrix).
So $A = CDC^{-1} = C0C^{-1} = 0$. $\square$

(c) Find a nonzero 2×2 matrix A which is similar to rA for every $r \neq 0$.

Solution

We know by Part (a) that all eigenvalues of A must be 0. So we need the characteristic polynomial to be $p(\lambda) = \lambda^2$. So we would take A to be of one of the forms $\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix}$. We try
$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. To be similar to $rA = \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}$
we need invertible C such that $C^{-1}AC = rA$. So we need $AC = rCA$. We take $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ and then
$$AC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} c_{21} & c_{22} \\ 0 & 0 \end{bmatrix} \text{ and}$$
$$rCA = r\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & c_{11} \\ 0 & c_{21} \end{bmatrix} = \begin{bmatrix} 0 & rc_{11} \\ 0 & rc_{21} \end{bmatrix}.$$

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So we need $\begin{bmatrix} c_{21} & c_{22} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & rc_{11} \\ 0 & rc_{21} \end{bmatrix}$ or $c_{21} = 0$ and $c_{22} = rc_{11}$. So we take $c_{11} = 1$, $c_{22} = r$, and (to try to simplify things, $c_{12} = 0$) so that $C = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/r \end{bmatrix}$. Then

$$C^{-1}AC = \begin{bmatrix} 1 & 0 \\ 0 & 1/r \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}$$

$$= \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = rA, \text{ as needed. } \square$$

(d) Show that $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ is similar to $-A$.

<u>Solution</u>

By Definition 5.4, "Similar Matrices," we need invertible 2×2 matrix C such that $C^{-1}AC = -A$; that is, $AC = -CA$. Let $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ so that

$$AC = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} c_{11}+c_{21} & c_{12}+c_{22} \\ -c_{21} & -c_{22} \end{bmatrix} \text{ and}$$

$$-CA = -\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -c_{11} & -c_{11}+c_{12} \\ -c_{21} & -c_{21}+c_{22} \end{bmatrix}.$$

This gives 4 equations in 4 unknowns:

$$\begin{aligned} c_{11}+c_{21} &= -c_{11} & &\text{or} & 2c_{11}+c_{21} &= 0. \\ c_{12}+c_{22} &= -c_{11}+c_{12} & & & c_{11}+c_{22} &= 0 \\ -c_{21} &= -c_{21} & & & 0 &= 0 \\ -c_{22} &= -c_{21}+c_{22} & & & c_{21}-2c_{22} &= 0 \end{aligned}$$

So consider

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \end{array}\right] \xrightarrow[R_3 \leftrightarrow R_4]{R_1 \to R_1 - R_2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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$$\xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 2 & 0 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_2]{R_1 \to R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So we need

$$\begin{aligned} c_{11} \quad + c_{22} &= 0 \\ -c_{21} \quad + 2c_{22} &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} c_{11} &= -c_{22}. \\ c_{21} &= 2c_{22} \\ c_{12} &= c_{12} \\ c_{22} &= c_{22}. \end{aligned}$$

So we take $C = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$. For C^{-1}, consider

$$[C|I] = \left[\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to -R_1} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right] = [I|C^{-1}].$$

So $C^{-1} = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$ and

$$C^{-1}AC = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} = - \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = -A, \text{ as required. } \square$$