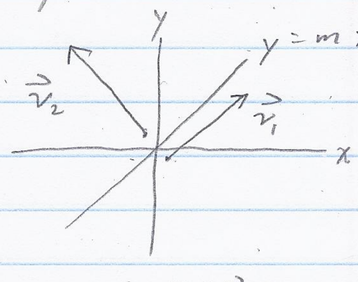
SECTION 5.2
EXERCISE #21

5.2.21 Find a formula for the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ that reflects vectors in the line $y = mx$.

Solution

We take a vector $\vec{v}_1$ along $y = mx$ and a vector $\vec{v}_2$ perpendicular to $y = mx$:



So we take $\vec{v}_1 = [1, m]$ and $\vec{v}_2 = [-m, 1]$.
Then we need T to map $\vec{v}_1$ to itself and map $\vec{v}_2$ to $-\vec{v}_2$; that is, $T(\vec{v}_1) = \vec{v}_1$ and $T(\vec{v}_2) = -\vec{v}_2$.
So $\lambda_1 = 1$ is an eigenvalue of T with $\vec{v}_1 = [1, m]$ as a corresponding eigenvector, and $\lambda_2 = -1$ is an eigenvalue of T with $\vec{v}_2 = [-m, 1]$ as a corresponding eigenvector. With A as the standard matrix representation of T we have

$$T\left(\begin{bmatrix}1\\m\end{bmatrix}\right) = A\begin{bmatrix}1\\m\end{bmatrix} = \begin{bmatrix}1\\m\end{bmatrix} \text{ and } T\left(\begin{bmatrix}-m\\1\end{bmatrix}\right) = A\left(\begin{bmatrix}-m\\1\end{bmatrix}\right) = \begin{bmatrix}m\\-1\end{bmatrix}.$$

So with $C = \begin{bmatrix}1 & m\\ m & -1\end{bmatrix}$, that is the columns of C are eigenvectors of A, and $D = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$, that is a diagonal matrix with diagonal entries as eigenvalues of A, then by Theorem 5.2, "Matrix Summary of Eigenvalues of A," we must have $AC = CD$. Now C is invertible and:

$$[C \mid I] = \left[\begin{array}{cc|cc} 1 & m & 1 & 0\\ m & -1 & 0 & 1\end{array}\right] \xrightarrow{R_2 \to R_2 - mR_1} \left[\begin{array}{cc|cc} 1 & m & 1 & 0\\ 0 & -1-m^2 & -m & 1\end{array}\right]$$

$$\xrightarrow{R_2 \to R_2/(-1-m^2)} \left[\begin{array}{cc|cc} 1 & m & 1 & 0\\ 0 & 1 & m/(1+m^2) & -1/(1+m^2)\end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - mR_2} \left[\begin{array}{cc|cc} 1 & 0 & 1-m^2/(1+m^2) & m/(1+m^2)\\ 0 & 1 & m/(1+m^2) & -1/(1+m^2)\end{array}\right] = [I \mid C^{-1}]$$

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so that $C^{-1} = \frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & -1 \end{bmatrix}$. Now $AC = CD$ implies $A = CDC^{-1}$ so

$$A = \begin{bmatrix} 1 & m \\ m & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & -1 \end{bmatrix}$$

$$= \frac{1}{1+m^2}\begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix}\begin{bmatrix} 1 & m \\ m & -1 \end{bmatrix} = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

So $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$= \boxed{\frac{1}{1+m^2}\begin{bmatrix} (1-m^2)x + 2my \\ 2mx + (m^2-1)y \end{bmatrix}.}$$

$\square$