SECTION 5.2
EXERCISE #23

5.2.23 Prove that if A and B are similar square matrices, then each eigenvalue of A has the same geometric multiplicity for A that it has for B. HINT: Use Exercise #22 which states: "Let A and C be $n \times n$ matrices, and let C be invertible. Prove that, if $\vec{v}$ is an eigenvector of A with corresponding eigenvalue λ, then $C^{-1}\vec{v}$ is an eigenvector of $C^{-1}AC$ with corresponding eigenvalue λ. Then prove that all eigenvectors of $C^{-1}AC$ are of the form $C^{-1}\vec{v}$, where $\vec{v}$ is an eigenvector of A."

Proof

If A and B are similar then $B = C^{-1}AC$ for some invertible C. Let λ be an eigenvalue of A of geometric multiplicity k and let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ be a basis for the eigenspace E_λ^A. Then by Exercise #22, $C^{-1}\vec{v}_1, C^{-1}\vec{v}_2, \dots, C^{-1}\vec{v}_k$ are eigenvectors of B with corresponding eigenvalue λ. [Notice that we have not shown these are all of the eigenvectors of B corresponding to λ nor have we even shown that all eigenvectors of B corresponding to λ are in $sp(C^{-1}\vec{v}_1, C^{-1}\vec{v}_2, \dots, C^{-1}\vec{v}_k)$.] By Exercise 2.2.37, the vectors $C^{-1}\vec{v}_1, C^{-1}\vec{v}_2, \dots, C^{-1}\vec{v}_k$ are independent. So a basis for the eigenspace E_λ^B for eigenvalue λ of matrix B. Since $sp(C^{-1}\vec{v}_1, C^{-1}\vec{v}_2, \dots, C^{-1}\vec{v}_k)$ is a subspace of E_λ^B then

$$\dim(sp(C^{-1}\vec{v}_1, C^{-1}\vec{v}_2, \dots, C^{-1}\vec{v}_k)) = k \le \dim(E_\lambda^B).$$

So $\dim(E_\lambda^A) = k \le \dim(E_\lambda^B)$ and the geometric multiplicity of eigenvalue λ for A is less than or equal to the geometric multiplicity of λ for B. Since $A = CBC^{-1}$ then we can similarly show that the geometric multiplicity of λ for B is less than or equal to the geometric multiplicity of λ for A to conclude the desired equality. ∎