SECTION 5.2
EXERCISE #25

5.2.25 Let $T : V \to V$ be a linear transformation of a vector space V into itself. Prove that, if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are eigenvectors of T corresponding to distinct nonzero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then the set $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$ is independent.

Proof

By Theorem 5.3, "Independence of Eigenvectors," $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly independent.

Consider $$r_1 T(\vec{v}_1) + r_2 T(\vec{v}_2) + \cdots + r_k T(\vec{v}_k) = \vec{0}. \quad (*)$$

Then $r_1(\lambda_1 \vec{v}_1) + r_2(\lambda_2 \vec{v}_2) + \cdots + r_k(\lambda_k \vec{v}_k) = \vec{0}$ since $\vec{v}_i$ is an eigenvector corresponding to eigenvalue λ_i for $i = 1, 2, \ldots, k$. Next,

$$(r_1\lambda_1)\vec{v}_1 + (r_2\lambda_2)\vec{v}_2 + \cdots + (r_k\lambda_k)\vec{v}_k = \vec{0}$$

and, since $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is independent, then $r_1\lambda_1 = r_2\lambda_2 = \cdots = r_k\lambda_k = 0$ by Definition 2.1, "Linear Dependence and Independence." Since λ_i is nonzero for $i = 1, 2, \ldots, k$ then it must be that $r_1 = r_2 = \cdots = r_k = 0$. That is (from $(*)$) $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$ is independent. ∎