SECTION 5.2

EXERCISE # 27

5.2.27 Exercise # 26 states that: "... the set $\{e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}\}$ where the λ_i are distinct, is independent in the vector space W of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ and having derivatives of all orders." Use this to prove that the infinite set $\{e^{kx} \mid k \in \mathbb{R}\}$ is an independent set in the vector space W described in Exercise # 26.

Proof

We use Definition 3.5, "Linear Dependence and Independence [in a Vector Space]." ASSUME there is a dependence relation

$r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_m \vec{v}_m = \vec{0}$ where some $r_i \neq 0$. (*)

and $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} \subset \{e^{kx} \mid k \in \mathbb{R}\}$. Then

$$r_1 e^{k_1 x} + r_2 e^{k_2 x} + \cdots + r_m e^{k_m x} = \vec{0}$$

(here "$\vec{0}$" represents the zero function on $\mathbb{R}$).

But by Exercise # 26, the set $\{e^{k_1 x}, e^{k_2 x}, \ldots, e^{k_m x}\}$ is an independent set and so $r_1 = r_2 = \cdots = r_m = 0$, CONTRADICTING equation (*). So there is no dependence relation on $\{e^{kx} \mid k \in \mathbb{R}\}$ and this set is independent, as claimed. ∎