SECTION 5.2

EXERCISE 5

5.2.5 Find the eigenvalues and corresponding eigenvectors of the matrix A and also find an invertible matrix C and a diagonal matrix D such that $D = C^{-1}AC$, where
$$A = \begin{bmatrix} -3 & 10 & -6 \\ 0 & 7 & -6 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution

For the eigenvalues, we consider the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 10 & -6 \\ 0 & 7-\lambda & -6 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$= (-3-\lambda)(7-\lambda)(1-\lambda)$ since $A - \lambda I$ is upper triangular (see Page 255 Example 4).

Set $p(\lambda) = (-3-\lambda)(7-\lambda)(1-\lambda) = 0$ and we have that the eigenvalues of A are $\boxed{\lambda_1 = -3,\ \lambda_2 = 1, \text{ and } \lambda_3 = 7.}$

For the corresponding eigenvectors, we consider the system of equations $(A - \lambda I)\vec{v} = \vec{0}$.

$\underline{\lambda_1 = -3}$

The system of equations $(A - (-3)I)\vec{v}_1 = \vec{0}$ yields

$$\left[\begin{array}{ccc|c} -3-(-3) & 10 & -6 & 0 \\ 0 & 7-(-3) & -6 & 0 \\ 0 & 0 & 1-(-3) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 10 & -6 & 0 \\ 0 & 10 & -6 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3/4]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 0 & 10 & -6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{R_1 \to R_1/10} \left[\begin{array}{ccc|c} 0 & 1 & -3/5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + \frac{3}{5}R_2} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

or $\begin{aligned} v_2 &= 0 \\ v_3 &= 0 \\ 0 &= 0 \end{aligned}$ or $\begin{aligned} v_1 &= v_1 \\ v_2 &= 0 \\ v_3 &= 0, \end{aligned}$

or with $r = v_1$ as a free variable $\begin{aligned} v_1 &= r \\ v_2 &= 0 \\ v_3 &= 0. \end{aligned}$

So the eigenvectors $\vec{v}_1$ associated with $\lambda_1 = -3$ are

$$\boxed{\vec{v}_1 = r\begin{bmatrix}1\\0\\0\end{bmatrix} \text{ where } r \in \mathbb{R},\ r \neq 0.}$$

$\underline{\lambda_2 = 1}$ The system of equations $(A-(1)I)\vec{v}_2 = \vec{0}$ yields

$$\left[\begin{array}{ccc|c} -3-(1) & 10 & -6 & 0\\ 0 & 7-(1) & -6 & 0\\ 0 & 0 & 1-(1) & 0\end{array}\right] = \left[\begin{array}{ccc|c} -4 & 10 & -6 & 0\\ 0 & 6 & -6 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} -4 & 4 & 0 & 0\\ 0 & 6 & -6 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \xrightarrow[R_2 \to R_2/6]{R_1 \to R_1/(-4)} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \text{ or } \begin{aligned} v_1 \quad - v_3 &= 0\\ v_2 - v_3 &= 0\\ 0 &= 0\end{aligned}$$

or $\begin{aligned} v_1 &= v_3\\ v_2 &= v_3\\ v_3 &= v_3\end{aligned}$ or with $s = v_3$ as a free variable $\begin{aligned} v_1 &= s\\ v_2 &= s\\ v_3 &= s.\end{aligned}$

So the eigenvectors $\vec{v}_2$ associated with $\lambda_2 = 1$ are

$$\boxed{\vec{v}_2 = s\begin{bmatrix}1\\1\\1\end{bmatrix} \text{ where } s \in \mathbb{R},\ s \neq 0.}$$

$\underline{\lambda_3 = 7}$ The system of equations $(A-(7)I)\vec{v}_3 = \vec{0}$ yields

$$\left[\begin{array}{ccc|c} -3-(7) & 10 & -6 & 0\\ 0 & 7-(7) & -6 & 0\\ 0 & 0 & 1-(7) & 0\end{array}\right] = \left[\begin{array}{ccc|c} -10 & 10 & -6 & 0\\ 0 & 0 & -6 & 0\\ 0 & 0 & -6 & 0\end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} -10 & 10 & 0 & 0\\ 0 & 0 & -6 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \xrightarrow[R_2 \to R_2/(-6)]{R_1 \to R_1/(-10)} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

or $\begin{aligned} v_1 - v_2 &= 0\\ v_3 &= 0\\ 0 &= 0\end{aligned}$ or $\begin{aligned} v_1 &= v_2\\ v_2 &= v_2\\ v_3 &= 0\end{aligned}$ or with $t = v_2$ as a free variable

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EXERCISE 5 (continued 2)

$v_1 = t$ so the eigenvectors $\vec{v}_3$ associated
$v_2 = t$ with $\lambda_3 = 7$ are
$v_3 = 0$.

$$\boxed{\vec{v}_3 = t\begin{bmatrix}1\\1\\0\end{bmatrix} \text{ where } t \in \mathbb{R},\ t \neq 0.}$$

By Theorem 5.2, "Matrix Summary of Eigenvalues of A of A," we let

$$D = \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix} = \begin{bmatrix}-3 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 7\end{bmatrix}$$

and

$$C = \begin{bmatrix}\vdots & \vdots & \vdots\\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3\\ \vdots & \vdots & \vdots\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 1 & 0\end{bmatrix}$$

where eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are chosen by letting $r = s = t = 1$. Notice that the by Theorem 5.3, "Independence of Eigenvectors," since $\lambda_1, \lambda_2, \lambda_3$ are distinct then matrix C is invertible ("nonsingular") and so C^{-1} exists. □